

# Can one hear the shape of a target zone?

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## Abstract

We develop an exchange rate target zone model with finite exit time and non-Gaussian tails. We show how the tails are a consequence of time-varying investor risk aversion, which generates mean-preserving spreads in the fundamental distribution. We solve explicitly for stationary and non-stationary exchange rate paths, and show how both depend continuously on the distance to the exit time and the target zone bands. This enables us to show how central bank intervention is endogenous to both the distance of the fundamental to the band and the underlying risk. We discuss how the feasibility of the target zone is shaped by the set horizon and the degree of underlying risk, and we determine a minimum time at which the required parity can be reached. We prove that increases in risk after a certain threshold can yield endogenous regime shifts where the “honeymoon effects” vanish and the target zone cannot be feasibly maintained. None of these results can be obtained by means of the standard Gaussian or affine models. Numerical simulations allow us to recover all the exchange rate densities established in the target zone literature.

**JEL Classification:** F31, F33, F45, C63

**Keywords:** Target zones; Non-stationary exchange rate dynamics; Dynamic mean-preserving spreads; Target zone feasibility; Spectral gap; Endogenous regime shifts.

## 1 Introduction

The exchange rate target zone literature pioneered by [Krugman \(1991\)](#) is based on a stochastic flexible price monetary model in continuous time. This literature highlights the role of market expectations concerning fundamentals in shaping exchange rate movements. Given its assumptions of perfect credibility, it implies that central bankers need only intervene marginally at the bounds of the target zone or allow honeymoon effects to automatically stabilize the exchange rate. The European Monetary System (EMS) and the Exchange Rate Mechanism (ERM), which existed from 1979 to 1999 (until participating countries adopted the Euro), provided a natural test bed for this theory.

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The target zone model is both well accepted theoretically and has provided the intellectual justification for a nominal anchor for monetary policy. However, there is scant empirical support for the validity of the framework. The U-shaped distribution within the target band and the negative correlation between the exchange rate and the interest rate differential implied by the Krugman model have found little counterpart in the data. In spite of this, the practice of using target zones continued through the 2000's with new member states joining the ERM-II target band and slowly adopting the Euro. It is conceivable that future new member states will go through the ERM process, making target zone modeling of current relevance.<sup>1</sup> Our purpose in this paper is to unpack target zone feasibility, while incorporating non-stationary dynamics and a rigorous measure of risk that captures the presence of non-Gaussian tails.

We make three main contributions. First, we show how such tails can emerge in the the fundamental dynamics of the exchange rate as a consequence of time-varying investor risk aversion. In our setting, the risk aversion of agents is subject to risk-on and risk-off shocks generating a time-varying coefficient of risk-aversion. These dynamics alter the idiosyncratic country-risk premium of small open economy, which destabilises the fundamental process via a sudden bonanza or sudden stop of capital flows during the target zone process. These tails are fully described by means of a definition of risk which corresponds to the dynamic equivalent of a mean-preserving spread. As is well known, risk and variance are not necessarily equivalent: variance is often used as a proxy for risk, but by construction it cannot capture tail risk in a random variable generated by a potentially non-Gaussian process.

Second, we explicitly consider non-stationary dynamics for a currency to exit a target zone, and show how the feasibility of the latter is shaped both by the finite time horizon and the degree of underlying risk. Solving for explicitly time-dependent dynamics also allows us to show how the exchange rate is continuously determined by the distance to the time horizon as well as its distance to the target bands. It turns out that the underlying dynamics are similar to the phenomena famously described by Kac (1966), where he asked whether one could “*hear the shape of a drum.*” In the case of exchange rates, in certain situations this can indeed happen, especially when the exchange rate is pushed to the sides of the target band by an additional external force: intuitively, this corresponds to the acoustic difference between striking a tense membrane (large shifts in risk aversion) versus a loose one. This is what we describe in our paper. This allows us to show how the central bank determines its intervention strategies by the degree to which it “feels” the presence of the target zone bounds, and depends critically on the degree of underlying risk and the band size.

Third, we show how large shocks to the investor risk aversion, leading to proportional increases in risk in the fundamental distribution, can potentially yield a regime shift once a certain risk threshold is crossed. This shift does not allow for honeymoon effects to happen anymore around the target zone bands, since the increase in risk destabilizes the exchange rate dynamics to the point that the target band is hit with excessive force, smooth-fitting procedures cannot be applied by central bank interventions and the target zone becomes untenable.

The standard case of exchange rate dynamics in a finite target zone with Gaussian-driven fundamentals is a simplified, limiting case of our model for which risk and variance are the same, and which fails to provide a palatable explanation for well-known exits such as ERM-I. Correctly specifying risk aversion shocks implies dynamics in which the exchange rate fundamental has a

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<sup>1</sup>As of writing this paper, Croatia, Bulgaria and Denmark are in the ERM-II target-zone. Croatia and Bulgaria intend to adopt the Euro whereas Denmark has a special opt-out clause from Euro adoption.

tendency to systematically escape its purely diffusive nature and move away from its expected value. As such, risk can be a destabilizing force which runs counter to the best efforts of a central bank trying to maintain a target zone. This may cause persistent and potentially one-sided deviations from central parity. Moreover, we show that the effect of risk is both nonlinear and discontinuous. For low risk, our dynamics are similar to the standard model. As risk increases, the exchange rate process is increasingly destabilized and requires a monotonically increasing minimum time for the target zone to be reached. However, once a critical threshold of risk is crossed, we observe a regime shift in which the minimum time suddenly drops down and the target zone effectively ceases to exist. The intuition behind this result is that the destabilizing part of the fundamental generated by external risk dominates over the diffusive part. This implies that the central bank has to widen the target zone bands in order to maintain control of the exchange rate. This is similar to target zone models which incorporate an endogenous devaluation risk, with one key difference. Our model does not rely on the distribution of intervention or the level of reserves, to generate this outcome, it's a direct consequence of an increase in external risk.

This ties directly to the characterization of feasibility of a target zone. Feasibility corresponds to the central bank being able to reach the set central parity with the agreed bands at the chosen time horizon. Our model shows that considering non-stationary dynamics is paramount in determining whether the chosen horizon is feasible: we characterize the minimum required time necessary for the parity to be reached. Any smaller time horizon chosen by the central bank would not be possible. In contrast, existing models assume away the problem by positing perfect feasibility and stationary dynamics. We then show how the model can fit a wide range of scenarios regarding feasibility and control, and we use Monte Carlo simulations to recover the different exchange rate densities presented by the established target zone literature.

## 2 Existing literature and motivations

The seminal paper by [Krugman \(1991\)](#) hinges on the assumption of perfect credibility of the target zone, which gives rise to a U-shaped distribution of the exchange rate. This implies that the exchange rate spends most of its time near the bands of the zone, as well as a negative relationship between the interest rate differential and exchange rate volatility. Given this “honeymoon effect”, the central bank only has to intervene marginally at the bands. The only source of risk in this model is the volatility of the Gaussian distribution. The theoretical predictions of the model have been shown not to hold empirically by [Mathieson et al. \(1991\)](#), [Meese and Rose \(1991\)](#) and [Svensson \(1991\)](#). This led to the development of so-called second-generation models, which relax Krugman’s assumptions across two dimensions, to allow for imperfect credibility of the target zone and for intramarginal intervention. The first dimension is studied by [Bertola and Caballero \(1992\)](#) and [Bertola and Svensson \(1993\)](#), who relax the notion of credibility and allow for time-varying credibility or realignment risk. They show that honeymoon effects disappear when there is a high probability of exchange rate revaluation. Furthermore, [Tristani \(1994\)](#) and [Werner \(1995\)](#) study endogenous realignment risk, and include mean-reverting fundamental dynamics.

Allowing for the possibility of realignment is a way of characterizing a riskier fundamental process, motivated by speculative attacks and constant realignment of the ERM currencies. This is achieved by using a diffusion process with jumps, as an *ad-hoc* way of thickening the tails of the distribution in order to better fit the data. The second dimension explored by second-generation models focuses on allowing the fundamental process to be controlled intramarginally, thus generating a hump-shaped distribution where the exchange rate spends most of its time around central parity. [Dumas and Delgado \(1992\)](#) and [Bessec \(2003\)](#), using controlled diffusion processes, show that the honeymoon effects are considerably weakened, putting into question the

necessity of a target zone when central banks intervene intramarginally. [Serrat \(2000\)](#) generalizes the target zone framework to a multilateral setting, and shows how spillovers from third-country interventions can increase conditional volatilities compared to free-float regimes. Moreover, this implies that exchange rate volatility does not need to be monotonically related to the distance to the target zone bands, which in turn can reduce honeymoon effects. [Bekaert and Gray \(1998\)](#) and [Lundbergh and Teräsvirta \(2006\)](#) test the implications of the second-generation models, and find mixed evidence with a slight tendency towards the intramarginal interventions hypothesis. [Lin \(2008\)](#) proposes a framework with an interesting analogy to our model, where the spot rate can be stabilized by imposing a target zone on the forward rate. This framework requires the setting a sequence of terminal maturity dates for the forward contracts, which generate forward-looking expectations that effectively endogenize the bands. This is similar to our model dynamics, which imply that the finite exit time from the target zone is akin to the maturity date of a forward contract, at which the spot rate is required to converge.

[Ajevskis \(2011\)](#) extended the basic target zone model to a finite termination time setting while maintaining the assumptions of the original model: it is the closest to our approach. [Ajevskis \(2015\)](#) extends his earlier contribution by allowing the exchange rate to follow a mean-reverting Ornstein-Uhlenbeck (OU) process and compares the difference in exchange rate-fundamental-target zone dynamics between the OU process and Brownian motion. He solves the stationary problem for the OU process but is unable to explicitly solve the non-stationary part of the process. Recently, [Studer-Suter and Janssen \(2017\)](#) and [Lera and Sornette \(2016, 2018 and 2019\)](#) find empirical evidence for the target zone model for the EUR/CHF floor target zone set by the Swiss National Bank between 2011 and 2015, the latter mapping the Krugman model to the option chain.

In particular, [Lera and Sornette \(2015\)](#) show how the standard Krugman model can hold in specific cases, such as the EUR/CHF target zone, because of a sustained pressure that continuously pushes the exchange rate closer to the bounds of the target zone, which the central bank tries to counteract. In this particular case, the sustained pressure stemmed from the Swiss Franc being used as a safe asset in the middle of the European crisis. This implies that there is a source of additional risk which is radically different from the diffusive nature of Gaussian noise. This risk destabilises the exchange rate fundamentals and creates an extra tendency to escape from its mean and move towards the boundary. [Rey \(2015\)](#) famously argued that the global financial cycles stemming from the United States generate additional risks for central banks targeting a nominal anchor. Additionally, [Gopinath and Stein \(2019\)](#) and [Kalemli-Özcan \(2019\)](#) show how US monetary policy shocks can affect the exchange rate of a country with minimal USD exposure because of the dominant nature of the USD as a trade currency. All of these examples represent possible sources of external risk that need to be included in the modeling of the fundamental process. Lastly, [Bauer et al. \(2009\)](#) shows how a model with heterogeneous agents and perfect credibility can create hump-shaped exchange rate distributions because of the contrasting forces between informed and uninformed traders. All pre-existing attempts at modeling fundamental risk involve either the variance of Gaussian noise or the addition of ad-hoc jumps, or by assuming deviations from rational expectations. In our paper, we show how all these resulting exchange rate densities can be recovered by a rigorous characterization of fundamental risk.

The paper is organised as follows. In [Section 3](#) we define possible interpretations and sources of external risk, in [Section 4](#) we extend the traditional stationary framework in order to include non-stationary dynamics, modeling the risk of the fundamental process by means of dynamic mean-preserving spreads. [Section 5](#) discusses the connection between risk, target zone width and feasibility. In [Section 6](#) we show the emergence of regime shifts once a critical threshold of risk is reached. [Section 7](#) explains the numerical methods employed in the simulations, and presents the results. [Section 8](#) presents the policy implications of our model, while [9](#) concludes and discusses an agenda for future research.

### 3 Risk aversion shocks, external sources of risk and mean-preserving spreads

In this paper we want to characterize a modern target zone mechanism in which the fundamental process can be destabilized by external risk factors, generating thick non-Gaussian tails in its distribution. Inclusion of these characteristics in the analysis is made necessary by the presence of risk-averse investors who have time varying risk-aversion modulated by the global financial cycle. Entering a target zone increases the capital market integration of the country in question which exposes countries' fundamentals to an increased share of global and regional risk factors.<sup>2</sup> Evidence from New Member States suggests that the magnitude of capital flows received may be very high even if the member state does not enter the target zone process for adopting the the Euro (Mitra, 2011).<sup>3</sup> In short, this framework allows us to consider additional fundamental risk arising from time-varying risk aversion generated by the global financial cycle when a currency enters a target zone.<sup>4</sup>

Let us start with the standard flexible-price monetary model of exchange rate as in Ajevskis (2011). The money demand function is given as

$$m_t - p_t = \theta_y y_t - \theta_i i_t + \epsilon \quad (1)$$

where  $m$  is log of the domestic money supply,  $p$  is log of the domestic price level,  $y$  is the log of domestic output and  $i$  is the nominal interest rate.  $\theta_y$  is the semi-elasticity of the money demand with respect to output whereas  $\theta_i$  is the absolute value of the semi-elasticity of money demand with respect to the domestic nominal interest rate and  $\epsilon$  is a money demand shock. The second block is given by the expression for the real exchange rate  $q$  which is defined as

$$q_t = X_t + p_t^* - p_t$$

where  $p^*$  is the log of the foreign price level. The third block of this model is the uncovered interest rate parity condition which in a linearised form is given by

$$\mathbb{E}dX_t = (i_t - i_t^*) - \eta_t \quad (2)$$

where  $\mathbb{E}dX_t$  is the expectation of the exchange rate conditional on information available till time  $t$  and  $i_t^*$  is the foreign interest rate. The target zone framework depends critically on the uncovered interest rate parity condition, with the currency in the target zone converging to the target nominal interest rate at time of exit to the currency union. The UIP condition requires risk-neutral preferences to hold. This is usually not the case when we are considering real world situations, as investors are generally risk-averse.  $\eta_t$  is a time-varying risk premium and is a consequence of risk-averse foreign investors who demand a higher compensation for holding home bonds.  $\eta_t$  is widely accepted to be dependent on investors' risk aversion. Risk

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<sup>2</sup>Fornaro (2020) finds that entering a currency union increases financial integration between member states. This is due to reduction of currency risk and the associated easing of external borrowing constraints, driven in part by loss of national monetary and fiscal autonomy. A target zone setting is a *quasi-currency union* with the chosen target zone band representing the range of expected fluctuations.

<sup>3</sup>This may be considered analogous to the index effects documented by Hau et al. (2010) for emerging market currencies.

<sup>4</sup>Destabilization of country fundamentals is also possible via shocks to dominant global currencies such as the Dollar, which may affect the price of risk for both the non-Dollar target currency and the target zone currency. See Rey (2015); Avdjiev et al. (2019) for more details.

aversion, however, is likely to change in time due to risk-on and risk-off shocks arising from global financial conditions. Let us consider that investors face a standard problem of consumption of two bonds, home and foreign, with concave utility  $U(c_t)$  discounted at  $\gamma$ .  $B_t^h$  is the holding of home (small open economy) bonds  $B_t^f$  is the holding of foreign bonds by a representative agent. Consumption and bond holdings in period  $t$  and  $t + 1$  are given by the problem

$$\begin{aligned} \max_{c_{t+1}, B_t^h, B_t^f} \quad & \sum_{t=0}^{\infty} \gamma^t U(c_t) \\ c_t = \quad & B_t^h + X_t B_t^f \\ \mathbb{E}[c_{t+1}] = \quad & (1 + i_t) B_t^h + \mathbb{E}[X_{t+1}](1 + i_t^*) B_t^f \end{aligned}$$

At any time  $t$ , if at the future time  $t + 1$  the agent's coefficient of relative risk aversion  $-cU''/U'$  were to be incremented by an amount  $\pm(\lambda \in \mathbb{R}^+)$  which can be either negative (risk-on) or positive (risk-off) with equal probability, yielding a new utility function  $\bar{U}$ .<sup>5</sup> Using the first-order conditions of the problem, this implies that the asset pricing kernel (the stochastic discount factor) will be given by

$$\gamma \frac{\bar{U}'(c_{t+1})}{U'(c_t)} = \gamma \frac{U'(c_{t+1})}{U'(c_t)} \Delta U'(c_{t+1}) = M_t \Delta U'(c_{t+1}),$$

where  $M_t$  is the pricing kernel without the change in risk aversion and  $\Delta U'(c_{t+1})$  is the change in curvature of the utility function due to the change in risk aversion. Note that this last term is also a random variable. As an example, if we assume CARA utility and a log-normal consumption process with mean  $\mu$  and variance  $s^2$ , this extra term is equal to  $e^{\pm\lambda(\mu - c_t - \lambda s^2/2)}$ , noting that a realized  $-\lambda$  in the utility functional implies an increase in risk aversion. More generally, if consumption of bonds is at two discrete time points but their evolution is continuous, this extra term is equivalent to the Radon-Nikodým derivative for the change of measure between the densities generated by the differently curved utility functions. The investors' pricing kernel is therefore

$$\gamma \frac{\bar{U}'(c_{t+1})}{U'(c_t)} = \frac{d\mathbb{Q}}{d\mathbb{P}} \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}$$

where  $\mathbb{Q}$  is the foreign martingale measure of the home bond under the original measure  $\mathbb{P}$ , and  $\tilde{\mathbb{Q}}$  is the foreign martingale measure under the new utility function. The modified UIP condition is then given by

$$\mathbb{E}\{dX_t\} \frac{(1 + i_t^*)}{(1 + i_t)} = \frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}}, \quad (3)$$

where the excess returns required to complete the no-arbitrage condition decreases with the investors' risk aversion, since  $\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}}$  increases with a realization of  $+\lambda$  (decreased risk aversion) and vice versa. This is equivalent to the modified UIP condition in (2), where the time-varying risk premium is dependent on the change in investor risk aversion. If we assume again log-normality of the foreign bond, since the change in risk aversion is equally likely on each side (each  $\pm\lambda$  is realized with probability 0.5), it's easily shown that that the new measure after the change in risk aversion is given by a Gaussian density identical to the pricing kernel without the curvature

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<sup>5</sup>This framework is equivalent to assuming heterogeneous investors, identical in everything except in risk aversion, where between  $t$  and  $t + 1$  each changes her own risk aversion to a specific amount, and the resulting  $\pm\lambda$  is the aggregate overall change in the representative utility function.



change, and an oscillating term that takes values  $\pm\lambda$  with equal probability, represented by a Bernoulli variable, that makes the overall process non-Gaussian. We note that the overall new measure  $d\tilde{\mathbb{Q}}/d\mathbb{P}$  is still a martingale but is not Gaussian, even assuming an underlying Gaussian process: the oscillation of the change in curvature of the utility function generates an extra term

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{1}{2} \left( e^{-(x+\lambda)^2/2} + e^{-(x-\lambda)^2/2} \right), \quad (4)$$

up to a normalization constant, which is exactly the perturbation of a Gaussian process by means of a Bernoulli variable in the drift. We therefore have a risk premium that is dependent on the oscillation of investors' risk aversion,  $\pm\lambda$  with equal probability. Using equations 1-4, we recover the standard flexible-price monetary model of the exchange rate as

$$\begin{aligned} \theta \mathbb{E} dX_t - \underbrace{\theta^y y_t + q_t + p_t^* + \theta i_t^* + \theta \eta_t - \epsilon + m_t}_{v_t} &= X_t \\ \underbrace{\hspace{10em}}_{f_t} \\ X_t &= \theta \mathbb{E}_t \{ dX_t \} + v_t + m_t \\ &= \theta \mathbb{E}_t \{ dX_t \} + f_t, \end{aligned} \quad (5)$$

where  $v_t$  is a money demand shock (velocity)  $m_t$  is money supply, usually assumed to be controlled by the central bank. The fundamental process for the exchange rate evolves according to  $df_t = dv_t + dm_t$ . The velocity includes all money demand variables, and therefore includes the varying risk premia from the modified UIP condition. It is commonly modeled as a driftless Brownian motion, but in order to include from (2) the perturbations caused by time-varying risk aversion given by (3) and (4) we augment it with the Bernoulli variable  $\lambda\mathcal{B}$ , where  $\mathcal{B}$  takes values  $\pm 1$  with probability 0.5. This allows us to represent the non-Gaussian dynamics required by the modified UIP condition, and allows us to write the fundamental process in absence of central bank interventions as

$$df_t = \lambda \mathcal{B} dt + dW_t, \quad f_{t=0} = f_0, \quad (6)$$

where  $dW_t$  is the standard Brownian motion and  $\mathcal{B}$  is a Bernoulli random variable obtaining values  $\{-1, 1\}$  each with probability 0.5 and  $\lambda \in \mathbb{R}^+$ . This non-Gaussian diffusion process, called the dynamic mean-preserving spread (DMPS) process, has been studied by [Arcand et al. \(2020\)](#). The oscillation of the Bernoulli random term generates a probability spread - an increase in risk - around the mean of the fundamental. For any  $f_0 \in \mathbb{R}$ , the fundamental dynamics do not affect the systematic average of  $f_t$ , but the extra random term  $\mathcal{B}$  introduces an extra tendency to shift away from  $f_0$ . The random term  $\lambda\mathcal{B}$  is not a drift that pushes systematically the fundamental away from the mean, but is rather a destabilizing force<sup>6</sup> that pushes probability from the center to the tails of the distribution. As such, the process  $f_t$  admits a stationary measure such that  $\lim_{t \rightarrow \infty} P(x, t | x_0, t_0) = P(x)$  which is the sum of two Gaussian distributions, each centered around  $\pm\lambda$ . By modulating  $\lambda$ , the resulting fundamental processes can be ranked unambiguously in terms of their risk, as the dynamic equivalent of the two [Rothschild and Stiglitz \(1970\)](#) integral conditions for increasing risk are satisfied. For all results concerning the process we refer to [Arcand et al. \(2020\)](#).

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<sup>6</sup>The term  $\lambda\mathcal{B}$  is indeed a force, being the derivative of the probabilistic potential of the process  $f_t$ .

Risk aversion shocks calibrated by  $\lambda$  in the velocity, therefore, cause an increase in risk in the fundamental that push probability away from the mean and generate non-Gaussian tails, whilst leaving the systematic average unchanged. What generates risk in the fundamental process is the size of the change in risk version, rather than the direction per se. As seen in 3, such shocks cannot be represented by Gaussian fluctuations. Uncertainty in the fundamental process is thus comprised of two parts: Gaussian fluctuations, as in the standard framework, and the destabilizing force that shifts probability to the tails and allows the fundamental process to escape normality. We can allow for a rescaling of the log-fundamental process by a sensitivity parameter  $\sigma < 1$ , and equation (6) can be written as:

$$df_t = \beta \mathcal{B} dt + \sigma dW_t, \quad (7)$$

where the Bernoulli variable  $\mathcal{B}$  now takes values  $\pm 1$ , where  $\beta = \lambda \sigma^2$  is the rescaled risk parameter.

We have identified the source of external exchange rate risk as exogenous changes in investor risk aversion: we note, however, the process (6) can be used to represent a variety of other sources of risk and destabilizing forces that cannot be represented reliably by Gaussian fluctuations, as well as to investigate the implications of such a modeling in a tractable way. In Appendix A we report alternative reduced form interpretations of  $\lambda$  as a source of destabilizing risk. Furthermore, in this paper we choose to focus on a fundamental process that remains stationary in distribution around its long-run level, here normalized to 0 without loss of generality. This is the case for most target zone cases. However, if the fundamental was substantially misaligned from its long-run level, then the choice of a mean-reverting process could be more appropriate. The analysis of this case is presented in Appendices E and F, where we fully solve both Ornstein-Uhlenbeck (O-U) and non-Gaussian, softly attractive dynamics. The latter can be of interest for researchers as an alternative to the O-U process, since it allows one to again escape Gaussianity and to model an ergodic process with light attraction towards its long-run level, whilst maintaining analytical tractability.

This also allows us to precisely characterize the interplay of diffusive fluctuations (variance) and destabilizing forces (risk, via changes in investor risk aversion): the tendency of external risk to shift the exchange rate away from the mean and towards the bounds of the zone is counteracted by the central bank's efforts to maintain the fundamental fluctuating around its mean. This is precisely what is argued by Lera and Sornette (2015). The standard Gaussian case is a limiting case for which the risk parameter is zero and there is no change in investor risk aversion. This is a more realistic characterization of fundamental risk, especially considering the influx of external risk given by global financial cycles. We show how the solution to the model is made up of two parts. The first is the time-independent stationary part, which corresponds to the behavior of the exchange rate at the time of entry in the target zone. The second is the transient part, which describes the sensitivity of the exchange rate to the distance to the bands, as a function of risk, band size and time to exit.

## 4 Target zone exit with a finite time horizon

We study the exchange rate equation derived from (5), which we write as

$$X_t = f_t + \frac{1}{\alpha} \mathbb{E} \{dX_t\}. \quad (8)$$

We allow explicitly time-dependent dynamics  $X_t = X(t, f_t)$ , and therefore study non-stationary behavior. At a fixed time  $T$  the spot exchange rate is set to exit the target zone and match



the target fundamentals. The absolute value of the semi-elasticity of money with the nominal interest rate,  $\theta$ , is always greater than unity: we rewrite it in the form  $1/\alpha$ , with  $0 < \alpha < 1$ , thus interpreting it with the dimension of a frequency (i.e.  $1/[\text{time unit}]$ ) which modulates the size of the forward-looking time window.

Using Itô calculus, Eq.(8) can be rewritten as:

$$\partial_t X(t, f) + \frac{\sigma^2}{2} \partial_{ff} X(t, f) + \beta \mathcal{B} \partial_f X(t, f) - \alpha X(t, f) = -\alpha f. \quad (9)$$

Note the presence of the additional term  $\partial_t X(f)$  in Eq.(22) which does not appear when one focuses only on stationary situations. The solution of (22) can be written as the sum of the time-independent stationary solution and the transient solution:

$$X(\tau, f) = X^*(\tau, f) + X_S(f). \quad (10)$$

Appendix B shows how the stationary solution of (10) is given by:

$$X_S(f) = \frac{1}{\cosh(\beta f)} \{A \mathcal{Y}_1(f) + B \mathcal{Y}_2(f) + Y_P(f)\}, \quad (11)$$

where we have:

$$\begin{cases} \mathcal{Y}_1(f) = \exp \left\{ +\sqrt{\left[\beta^2 + \frac{2\alpha}{\sigma^2}\right]} f \right\}, \\ \mathcal{Y}_2(f) = \exp \left\{ -\sqrt{\left[\beta^2 + \frac{2\alpha}{\sigma^2}\right]} f \right\}, \\ Y_P(f) = \frac{2\alpha(f(2\alpha + \beta^2(1 - \sigma^2)) \cosh(\beta f) + 2\beta\sigma^2 \sinh(\beta f))}{(2\alpha + \beta^2(1 - \sigma^2))^2} \end{cases} \quad (12)$$

In Eqs. (11) and (12) the pair of constants  $A$  and  $B$  can be determined by smooth fitting at the bounds  $\underline{f} = -\bar{f}$ :<sup>7</sup>

$$\partial_f X_S(f) |_{f=\underline{f}} = \partial_f X_S(f) |_{f=\bar{f}} = 0. \quad (13)$$

The two constants of integration  $A$  and  $B$  can be obtained in closed form but their expression is lengthy and is therefore omitted. An illustration of the stationary solution (11) is presented in Figure 1, which also shows how an increase in the riskiness  $\beta$  of the fundamental prompts the (stationary) exchange rate to behave more independently of the dynamics of the fundamental. At high levels of  $\beta$ , the exchange rate dynamics are driven mostly by the risk and depend less on fundamentals, especially around the bounds, as represented by the steepening of the central slope. In this figure,  $\bar{f} = 10\%$  and we assume a quasi-daily time step for the expectation  $\alpha = 0.8$ . Our parametrization of  $\alpha = 0.8$  corresponds to a case of fast agent updating, which is similar to the case studied by Ferreira et al. (2019) and Coibion and Gorodnichenko (2015). Changing the  $\alpha$  to a lower fundamental updating frequency will reduce the sensitivity of the exchange rate to the fundamentals.

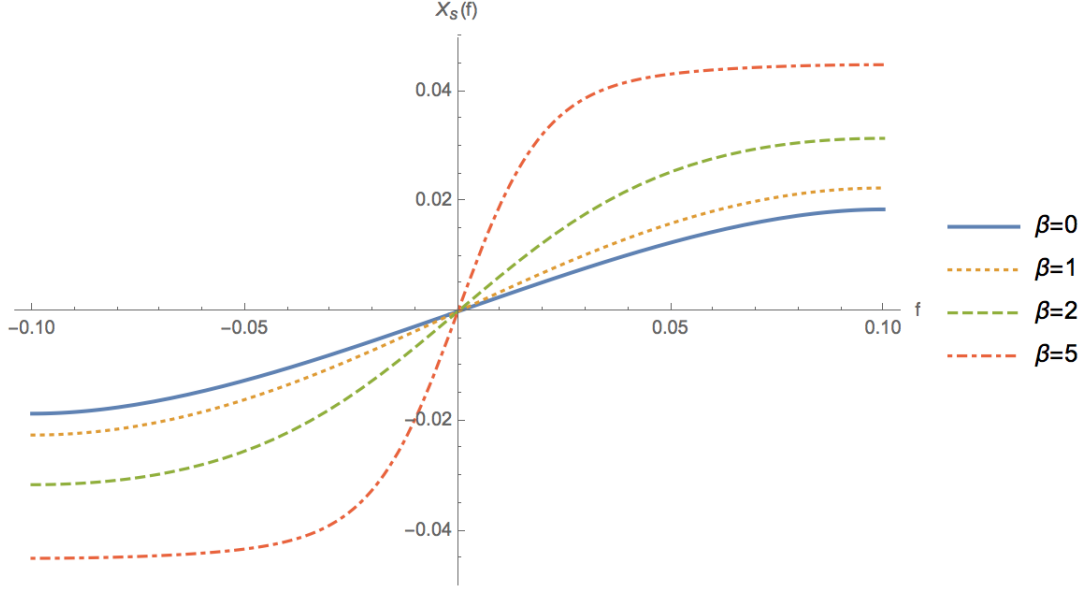
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<sup>7</sup>For simplicity we focus our attention to targets zones symmetric with respect to  $f = 0$ , although the results hold for general bounds. To see this, notice that the general solution is unaffected by the bounds, which enter the particular solution only via the scalar quantities  $A$  and  $B$ . We continue using a symmetric band  $[-\bar{f}, \bar{f}]$  for clarity of exposition.

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**Figure 1** Effect of varying  $\beta$  on stationary exchange rate dynamics

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We now turn to the transient dynamics. At a given time horizon  $t = T$ , we fix the predetermined exchange rate  $X(T, f) = 0$ . In terms of the backward time  $\tau = T - t$ , we write the transformation  $X^*(\tau, f) = Y^*(\tau, f) / \cosh(\beta f)$ . The time-dependent partial differential equation we need to solve is therefore given by:

$$\partial_\tau Y^*(\tau, f) - \frac{\sigma^2}{2} \partial_{ff} Y^*(\tau, f) + \left[ \frac{\beta^2}{2} + \alpha \right] Y^*(\tau, f) = 0. \quad (14)$$

with boundary conditions given by:

$$\begin{cases} [\partial_f Y^*(\tau, f) - \beta \tanh(\beta f) Y^*(\tau, f)]_{f=\underline{f}} = 0, \\ [\partial_f Y^*(\tau, f) - \beta \tanh(\beta f) Y^*(\tau, f)]_{f=\bar{f}} = 0. \end{cases} \quad (15)$$

We express the solution  $Y(\tau, f)$  as  $Y^*(\tau, f) = \phi(\tau)\psi(f)$ , and proceed to solve this equation by separation of variables and expansion over the basis of a complete set of orthogonal eigenfunctions. Sturm-Liouville theory allows us to state that on the interval  $[-\bar{f}, +\bar{f}]$ , one has a complete set of orthogonal eigenfunctions  $\psi_k(f)$  satisfying Eq.(13), namely:

$$\psi_k(f) = \sin \left( \frac{\sqrt{2}\Omega_k}{\sigma} f \right) \in [\underline{f}, \bar{f}], k = \mathbb{N}^+, \quad (16)$$

where each eigenvalue  $\Omega_k$  solves the transcendental equation:

$$\frac{\sqrt{2}\Omega_k}{\sigma} \cot \left( \frac{\sqrt{2}\Omega_k}{\sigma} \bar{f} \right) = \beta \tanh(\beta \bar{f}). \quad (17)$$

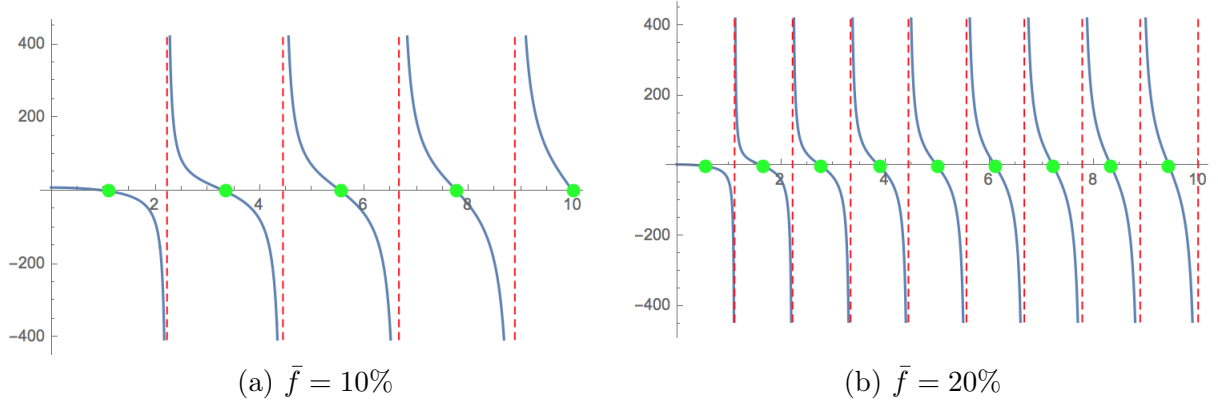
Furthermore, the eigenvalues are real and span a discrete spectrum:

$$\{\Omega_k\} := \{\Omega_k(\beta, \bar{f})\}, k \in \mathbb{N}^+.$$

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**Figure 2** Target band and spectrum

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Graphical illustration of the solution of equation (17), showing the effect of varying  $\bar{f}$  on the spectrum  $\Omega_k$ .

---

and can therefore be ordered as:

$$\Omega_1(\beta, \bar{f}) < \Omega_2(\beta, \bar{f}) < \dots$$

For any  $k \in \mathbb{N}^+$ , the corresponding  $\Omega_k(\beta, \bar{f})$  solves the transcendental equation (17), and has to be calculated numerically. For a general  $\beta > 0$ , one observes that the successive eigenvalues are not evenly spaced, and display a distance which decreases in  $k$ . The spectrum is controlled by the width of the target zone  $\bar{f}$ : the wider the band, the smaller the separation. The spectrum and its relationship with the target band size are illustrated in Figure 5. Observe also that in the limit  $\beta = 0$ , one straightforwardly verifies that from Eq.(16) one obtains the evenly spaced set  $\Omega_k(0, \bar{f}) = (2k + 1) \frac{\pi}{2\bar{f}}$ .

The development of the non-stationary solution  $X^*(\tau, f)$  over the complete set  $\{\psi_k(f)\}$  enables one to finally write the full expansion as:

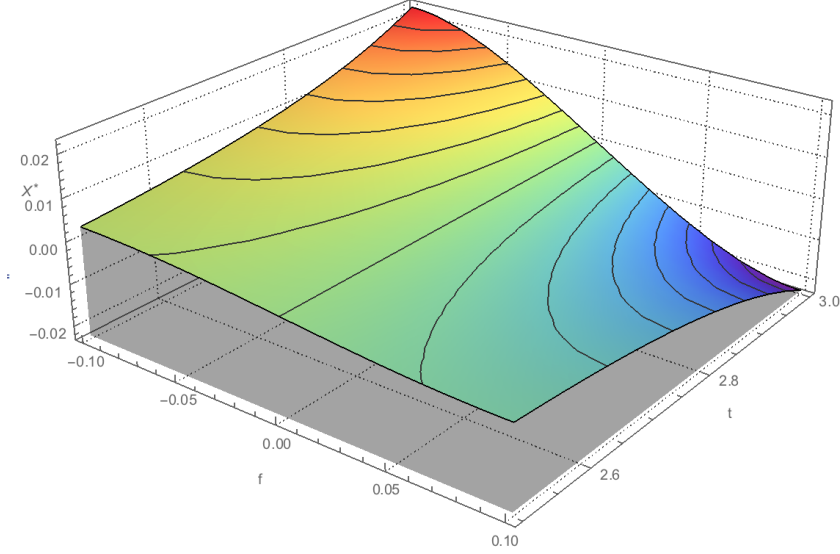
$$\begin{aligned}
 X^*(\tau, f) &= X^*(T - t, f), \quad t \in [0, T] \\
 &= \frac{1}{\cosh(\beta f)} \sum_{k=1}^{\infty} c_k \exp [-(\Omega_k^2 + \rho)(T - t)] \sin \left[ \frac{\sqrt{2}\Omega_k}{\sigma} f \right] \\
 c_k &= -\frac{1}{\bar{f}} \int_{-\bar{f}}^{+\bar{f}} X_S(f) \sin \left[ \frac{\sqrt{2}\Omega_k}{\sigma} f \right] df \\
 \rho &= \left[ \frac{\beta^2}{2} + \alpha \right].
 \end{aligned} \tag{18}$$

The full derivation is reported in Appendix B. When  $t = T$ , from Eq.(19), by construction of the Fourier coefficients  $c_k$ , we have  $X^*(0, f) = -X_S(f)$  and so  $X(T, f) = X^*(0, f) + X_S(f) = 0$  thus reaching the required fixed parity. An illustration of the non-stationary exchange rate dynamics, as well as the overall transition dynamics throughout the time interval  $[0, T]$ , is presented in Figure 3. This solution allows one to express the movements of the exchange rate via a weighted sum of its stationary behavior, its distance to the exit time and the distance between its value at any time  $t$  and the target band. The eigenvalues modulate the frequency of both fundamental and exchange rate movements within the band. The Fourier coefficients  $c_k$

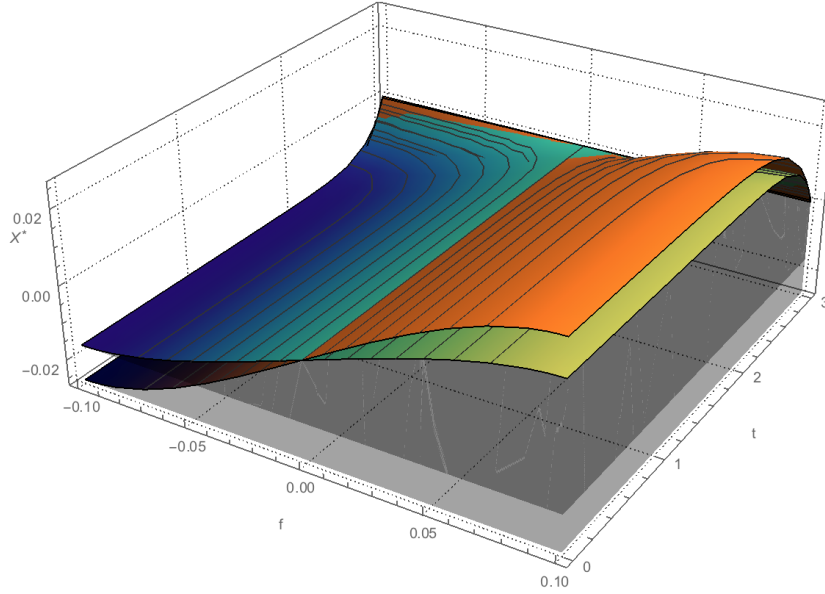
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**Figure 3** Non-stationary dynamics

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(A)



(B)

This figure shows the evolution of  $X(T-t, f)$  of the non-stationary dynamics in the target zone. Panel (A) shows the behavior of the time-dependent part: we assume a target zone which has been set to  $T = 3$  years, with  $\beta = 1$  for a given set of fundamentals. For the sake of brevity we truncate the figure towards the end of the target zone to effectively illustrate the non-stationary dynamics. Panel (B) shows the full dynamics for an increase in risk. Here we have assumed a target band symmetric around zero, i.e.  $\bar{f} = 10\% = -\underline{f}$ . We also assume  $\alpha = 0.8$ . We truncate the eigenfunction expansion at 50. The second panel illustrates the change in dynamics from  $\beta = 0$  (Gaussian) to  $\beta = 5$ .

---

represent the impact of the size of the target band in the overall dynamics, via their weight on the infinite series of frequency components (the “harmonics” of the exchange rate path). Loosely speaking, this formulation of the solution allows one to describe the sensitivity of the exchange rate to the distance to the target band. Once the eigenvalues and the eigenfunctions are known, as famously asked by Kac (1966), “if one had perfect pitch”, one would be able to “hear” the shape of the target zone.<sup>8</sup>

This formulation of the solution allows us to uncover the unique nature of the smooth-pasting conditions: the exchange rate process is not reflected at the bounds in the probabilistic sense, since this would have been modeled as a zero derivative condition on the transition probability density function. We are in the presence of “soft” boundaries, where the central bank interventions are determined by the interplay of the distance of the exchange rate to the bounds as well as the tendency of the fundamental to hit them (the risk): this is what is implied by the eigenfunction expansion of the solution. This allows us to “endogenize” the bands: because of the presence of expectations in the exchange rate equation (22), we have a second-order term which allows us to solve the equation in its Sturm-Liouville form and eigenfunction expansion. The Fourier coefficients modulate the sensitivity of the exchange rate to the distance to the band, allowing for the central bank to intervene whenever the fundamental is “felt” to be approaching the bounds. This “feeling” is in fact a direct translation of how much the fundamental tends to escape and how much the central bank needs to intervene marginally or intramarginally: it is a direct consequence of the presence of expectations in the exchange rate equation. In other words, the higher the tendency to hit the bounds, the greater is the likelihood that the central bank will actually intervene intramarginally, with increasingly less weight placed on the actual position of the fundamental within the band. One can therefore see that the higher is the risk (the fundamental’s tendency to escape from its central position), the more the central bank intervenes intramarginally. The same applies when the target band shrinks. The standard Krugman framework applies when the fundamental is a pure Brownian motion and the central bank only intervenes marginally. Note that this phenomenon is directly a consequence of our rigorous characterization of fundamental risk. In Section 7 we show how the model can replicate the different exchange rate densities under different assumptions of feasibility and intervention.<sup>9</sup> It is also worth noting that this framework potentially allows for the existence of *de jure* and *de facto* bands, as noted by Lundbergh and Teräsvirta (2006): if the *de jure* band is large, expectations over the magnitude of risk may react to a narrower *de facto* band. This is a phenomenon commonly observed in most ERM countries.

## 5 Risk, target width and feasibility: the role of the spectral gap

Here we discuss the interplay between the risk parameter  $\beta$ , the size of the target band  $[-\bar{f}, +\bar{f}]$  and the feasibility of the time horizon  $T$  at which to reach the target zone. We first note that at the initial time  $t = 0$ , from Eq.(19) we have  $X^*(T, f) \approx 0$  and therefore  $X(0, f) = X^*(T, f) + X_S(f) \approx X_S(f)$ . Since  $\Omega_1(\beta, \bar{f}) < \Omega_2(\beta, \bar{f}) < \dots$ , one can approximately write:

$$X(T, f) \simeq X_S(f) + \mathcal{O}\left(e^{-(\Omega_1^2 + \rho)T}\right).$$

<sup>8</sup>Note that the time-independent part of the problem is a one-dimensional Neumann problem on the boundary  $\partial D = [\underline{f}, \bar{f}]$

$$\begin{cases} \Delta f + \Omega f = 0 \\ \nabla f|_{\partial D} = 0, \end{cases}$$

which is exactly the problem of finding the overtones on a vibrating surface.

<sup>9</sup>For additional information, see Figure 2 in Crespo-Cuaresma et al. (2005)

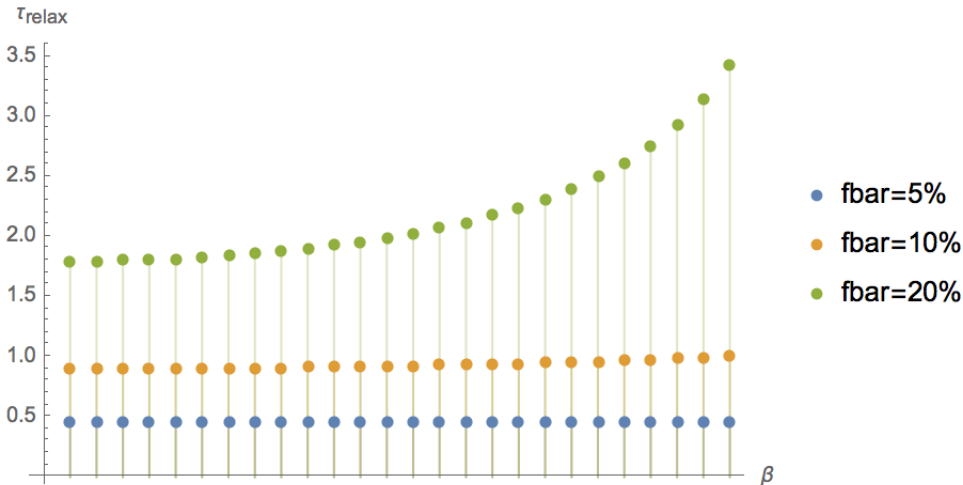
While for the exact solution we should have  $X(T, f) = X_S(f)$ , one sees immediately that  $X(T - t, f) = X_S(f) + X^*(T - t, f)$  with  $X^*(T - t, f)$  given by Eq.(19) nearly matches the exact solution, provided we have an horizon interval  $T \gtrsim t_{\text{relax}}$  where  $t_{\text{relax}} := (\Omega_1^2 + \rho)^{-1}$  is the characteristic relaxation time of the exchange rate process. This provides a validity range for the non-stationary dynamics given by the expansion Eq.(19).

Hence, at time  $t = 0$ , the required initial probability  $X_S(f)$  law is reached only for a large enough time horizon  $T \gtrsim t_{\text{relax}}$ . This now enables us to link the non-stationary dynamics of  $X^*(t, f)$  to the **feasibility** of the target zone: the relaxation time  $\tau_{\text{relax}}$  determines the *minimum time interval* for which a feasible target zone may be maintained. The larger  $\beta$  (the risk of the fundamental, stemming from larger shifts in agents' risk aversion), the greater is the tendency of the fundamental to escape from its mean; the authorities need therefore to maintain the target zone for a longer minimum duration. An increase in risk, for a given  $\bar{f}$ , implies that the target zone would have to be set for a longer horizon  $T$  to be feasible. Alternatively, for a given risk  $\beta$ , an increase of the target zone width  $\bar{f}$ , requires a longer minimal  $T$  implementation to ensure the overall feasibility of the policy. In other words, the central bank has to impose that the time horizon  $T$  is at least as large as the relaxation time  $t_{\text{relax}}$ .

An intuitive interpretation of the relaxation time in this framework is to understand  $t_{\text{relax}}$  as the characteristic elapsed time required to “feel” the first effects of the home central bank's actions aimed at reducing fluctuations of the exchange rate, compared to a free float. The bank's actions may be then viewed as a *de facto* reduction of the target zone band over time, whilst the *de jure* band remains unchanged. A possible implication would be that  $t_{\text{relax}}$  would be the minimum time for agents to update their priors accurately, generating self-fulfilling expectations that create the honeymoon effect.

The inverse of the relaxation time is determined by the **spectral gap**, which is the distance between 0 and the smallest eigenvalue. We therefore have the relationship  $(t_{\text{relax}})^{-1} = (\Omega_1^2 + \rho)$ . The spectral gap controls the asymptotic time behaviour of the expansion given by (19), and it is continuously dependent on risk  $\beta$  and band  $\bar{f}$ . This relationship is illustrated in Figure 4.

**Figure 4** Interaction between  $\beta$  and  $\bar{f}$



*Note:* This figure shows the interaction of varying risk ( $\beta$ ) and varying the band size ( $\bar{f}$ ). An increase in risk, for a given  $\bar{f}$ , implies that the lowest eigenvalue  $\Omega_1$  falls (Panel (A)). The inverse of this value controls the  $t_{\text{relax}}$ .

Let us now study analytically the behaviour of the solution  $\Omega_1$  of the transcendental Eq.(17). Writing  $z = \sqrt{2}\Omega_1\bar{f}$ , Eq.(17) implies that the product  $\beta\bar{f}$  is the determinant of the amplitude of



$\Omega_1$ . An elementary graphical analysis enables one to conclude that two limiting situations can be reached:

$$\begin{cases} \beta \bar{f} \ll 1 & \Rightarrow z \lesssim \frac{\pi}{2} & \Rightarrow \Omega_1 \lesssim \frac{\pi}{2\bar{f}} & \Rightarrow t_{\text{relax}}^{-1} \lesssim \left[ \frac{\pi}{2\bar{f}} \right]^2 + \frac{\beta^2}{2} + \alpha, \\ \beta \bar{f} \gg 1 & \Rightarrow z \gtrsim \pi & \Rightarrow \Omega_1 \gtrsim \frac{\pi}{\bar{f}} & \Rightarrow t_{\text{relax}}^{-1} \gtrsim \left[ \frac{\pi}{\bar{f}} \right]^2 + \frac{\beta^2}{2} + \alpha. \end{cases}$$

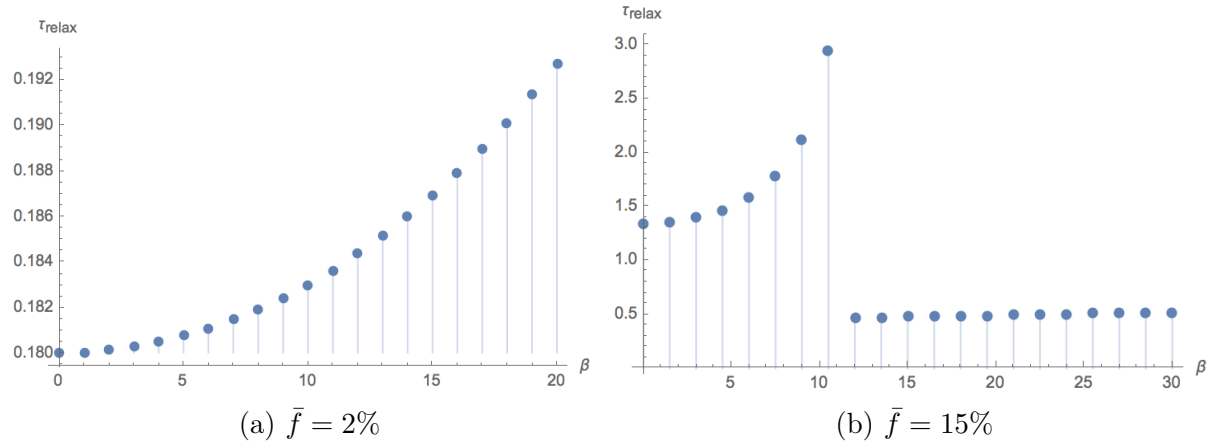
and therefore:

$$\frac{1}{\left[ \frac{\pi}{\bar{f}} \right]^2 + \frac{\beta^2}{2} + \alpha} \leq t_{\text{relax}} \leq \frac{1}{\left[ \frac{\pi}{2\bar{f}} \right]^2 + \frac{\beta^2}{2} + \alpha}. \quad (19)$$

Eq.(19), together with Figure 4 shows how an increase in risk  $\beta$  affects  $t_{\text{relax}}$  more strongly when the exchange rate is allowed to float in a wider band width  $\bar{f}$ .

## 6 Risk, target band and regime shifts

**Figure 5** Risk, target band and regime shifts



Regime shift and eigenvalue jump as a function of risk, for different target bands

An unique phenomenon that emerges when considering non-converging drifts, and in particular our specification, is the emergence of a *regime shift*. Figure 5(b) shows that for a large enough target band, after a threshold level in  $\beta$ , the relaxation time suddenly jumps to a much lower value and remains almost constant (though very slowly increasing) for further increases in risk. This effect happens because when the tendency  $\beta$  of the noise source driving the fundamental reaches and surpasses a certain level, the destabilizing risk component in the noise source overcomes the diffusion. The force  $\beta\mathcal{B}$  in the mean-preserving spread becomes the main driver of the stochastic process driving the fundamental, and therefore  $f_t$  becomes a process with a tendency to escape from its mean that is stronger than the tendency to diffuse around its central value. While this may look like a sudden emergence of supercredibility, it is in fact the opposite: the target zone cannot be feasibly held. This implies that the fundamental process escapes its initial position with such force that it hits the band at every  $dt$ , and interventions become almost continuous. Furthermore, smooth-pasting conditions cannot be applied anymore. The central bank will have to either increase the size of the band or to allow the spot rate to float freely. This has a direct implication for honeymoon effects: Appendix C shows how, after a threshold level

of risk has been surpassed, the smooth fitting procedure at the boundaries cannot be applied, and hence honeymoon effects when the fundamental approaches the band become unobtainable. This implies that a high level of risk denies a central bank monetary autonomy up until the moment of entering the currency zone. This phenomenon is illustrated in Figure 6.

Consider now the smooth-pasting conditions (15): one can separate the contribution of the eigenfunction to the one given by the probability spread and obtain:

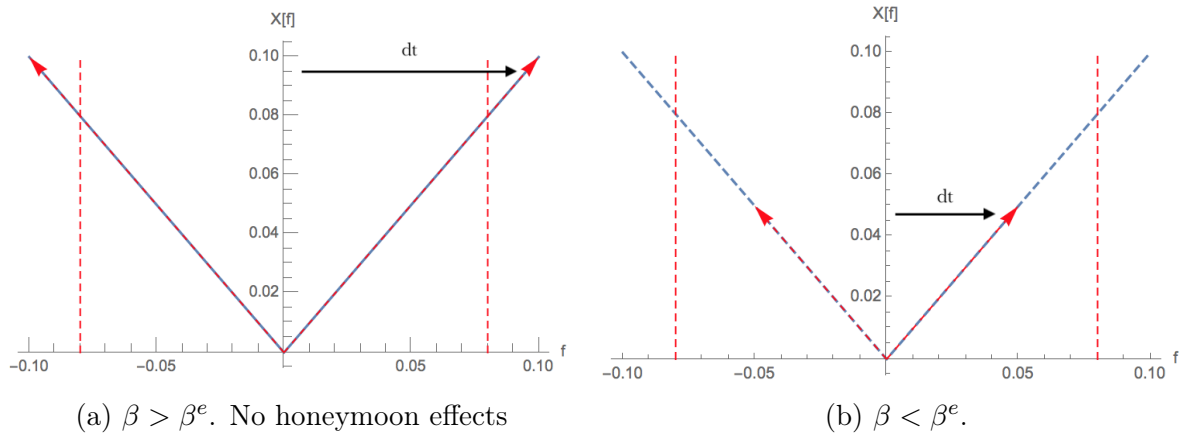
$$\begin{aligned} \frac{\partial_f \sin\left(\frac{\sqrt{2}\Omega_k}{\sigma} f_t\right)}{\sin\left(\frac{\sqrt{2}\Omega_k}{\sigma} f_t\right)} - \beta \tanh(\beta f_t) &= 0 \\ \Downarrow \\ \frac{\partial_f \text{EIG}(\Omega_k, f_t)}{\text{EIG}(\Omega_k, f_t)} - \text{MPS}(\beta, f_t) &= 0. \end{aligned} \quad (20)$$

The first term is a total sensitivity term, closely related to the elasticity of the eigenfunction with respect to the fundamental, and it represents the overall variation of the exchange rate with the fundamental. The second term represents the increase in risk, as well as the destabilizing component that represents the tendency of the fundamental to hit the target bands. The solution of this equation yields the spectrum  $\{\Omega_k\}$ , for  $k = \mathbb{N}^+$ . The difference of the two terms represents the residual tendency of the home country fundamental to avoid converging to the target fundamental. The spectral gap, therefore, represents the intensity of the probability spread. The regime shift will happen at a threshold value  $\beta^e$ , only obtainable numerically, for which the spectral gap will suddenly jump upwards: the destabilizing force has dominated over the diffusive part and the first eigenvalue jumps higher. The oscillating part of the expansion increases in frequency, and the time-dependent exponential decay increases in speed. A graphical illustration is shown in Figure 7: one can easily show that the lower bound for the threshold  $\beta^e$  is given by  $1/\bar{f}$ . This allows one to uncover the close relationship between the regime shift and the size of the target band. This regime shift **cannot** occur with a Gaussian process or with mean-reverting dynamics.

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**Figure 6** Risk threshold, distance to the bands and honeymoon effects

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Large risk shocks vs. diffusion-driven regimes

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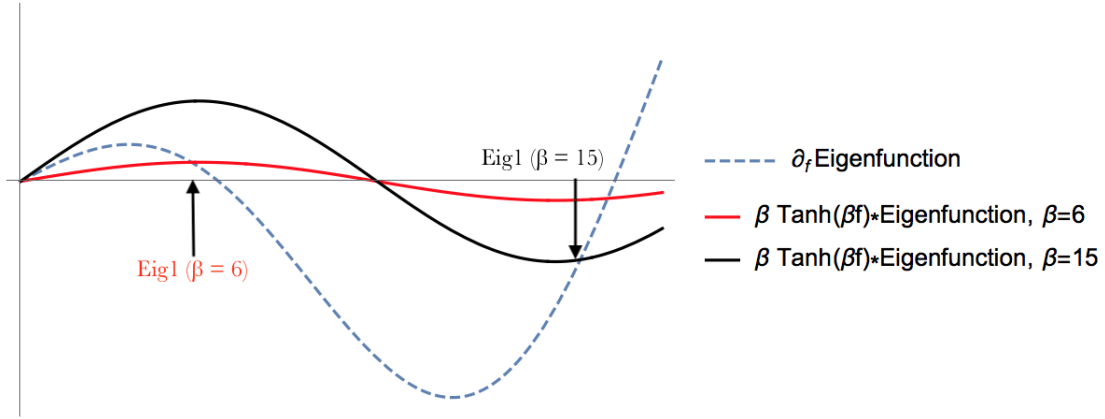
In the diffusion-driven regime (characterised by a relatively low  $\beta < 1/\bar{f}$ ), one observes that an increase of risk implies a decrease in sensitivity, since  $t_{relax}$  is increasing. This may seem coun-

terintuitive: but it must be remembered that at time  $t = 0$ , the initial condition is the stationary solution of the central bank-controlled diffusion for the given risk. Increasing  $\beta$ , therefore, is likely to load the stationary probability mass accumulated in the vicinity of the target zone boundaries. Escape from this stationary state by bank action becomes more difficult, ultimately leading to an increase of  $t_{relax}$ . Conversely, in high risk regimes where  $\beta > 1/\bar{f}$  and where the destabilizing dynamics dominate, the boundaries of the target zone are systematically hit by the fundamental. In this situation, the central bank will intervene almost entirely intramarginally regardless of whether the fundamental is actually close to the bands, since honeymoon effects cannot exist anymore. This allows, in Eq.(12), for a sudden reduction of the probability mass located at the bounds, and this generates the sharp drop of  $t_{relax}$ . In other words, the band implicitly ceases to exist and the central bank operates effectively in an infinitesimally narrow band. This provides new insight into target zone feasibility: if risk is too high, exchange rate expectations are no longer anchored to the band and the effectiveness of central bank intervention is greatly reduced. What the central bank could do is therefore either (i) to reduce risk, which in practice is often infeasible, or (ii) to increase the size of the target zone which itself is bounded by the free-float exchange rate volatility. The new size of the band would have to be large enough for this new target zone to be “heard”.

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**Figure 7** Risk and eigenvalue jump

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*Note:* Regime change For  $\beta = 15$ . The force  $\beta \tanh(\beta f)$  (black curve) overcomes the diffusion component and generates the first eigenvalue jump. For  $\beta = 6$  (red curve), the regime has not yet shifted. Here  $\bar{f} = 0.1$ ,  $\sigma = 1$ ,  $\alpha = 0.8$ .

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We can therefore also connect the threshold  $\beta_e$  at which the regime shift occurs to complete factor market integration: for lower levels of  $\beta$ , the home fundamental exhibits an idiosyncratic component anchored to its original dynamics that is stronger than its tendency to converge to the target fundamental. Once this component is overcome, the target zone ceases to exist and the currency starts floating. This may also help explain why countries with a high level of capital integration with the target currency may have higher costs in maintaining a target zone. One implication of the suddenness of the regime shift is that the relationship between capital integration and the duration of the target zone is non-monotonic. This is precisely what [Lera and Sornette \(2015\)](#) illustrate with the case of the Swiss Franc floor between 2011-2015.

## 7 Numerical simulations

We simulate central bank intervention by means of a symmetrized Euler scheme for stochastic differential equations. Since the original problem is a one-dimensional Neumann problem on the boundary  $\partial D = [-\bar{f}, \bar{f}]$ , the regulated SDE can be written as:

$$f_t = \int_0^t b(f_s)ds + \sigma \int_0^t dW_s + \int_0^t \gamma(f_s)ds,$$

where  $b(f_s)$  is the nonlinear drift and  $\gamma(\cdot)$  is the oblique reflection of the process on the boundary  $\partial D$ . This is the equivalent of the interventions, and we assume that for the unit vector field  $\gamma$  there exists a constant  $c$  so that  $\gamma(x) \cdot \vec{n}(x) \geq c$  for all points  $x$  on the boundary  $D$ . This can be interpreted as assuming bounded interventions. We use a regular mesh  $[0, T]$  for the numerical simulation, for which the weak error is of order 0.5 when the reflection is normal (i.e.  $\gamma = \vec{n}$ ), which is our case. We choose this method in order to obtain consistent Monte Carlo simulation of the resulting densities. The algorithm starts with  $f_0 = 0$  and for any time  $t_i$  for which  $f_{t_i} \in D$  we have for  $t \in \Delta t = t_{i+1} - t_i$  that:

$$F_t^{N,i} = f_{t_i}^N + \hat{b}(f_{t_i}^N)(t - t_i) + \sigma(W_t - W_{t_i}),$$

as in the standard Euler-Maruyama scheme, and the nonlinear drift  $b(\cdot)$  is approximated with a second-order stochastic Runge-Kutta method. If  $F_{t+1}^{N,i} \notin \partial D$ , then we set:

$$f_{t+1}^N = \pi_{\partial D}^\gamma(F_{t+1}^{N,i}) - \gamma(F_{t+1}^{N,i}),$$

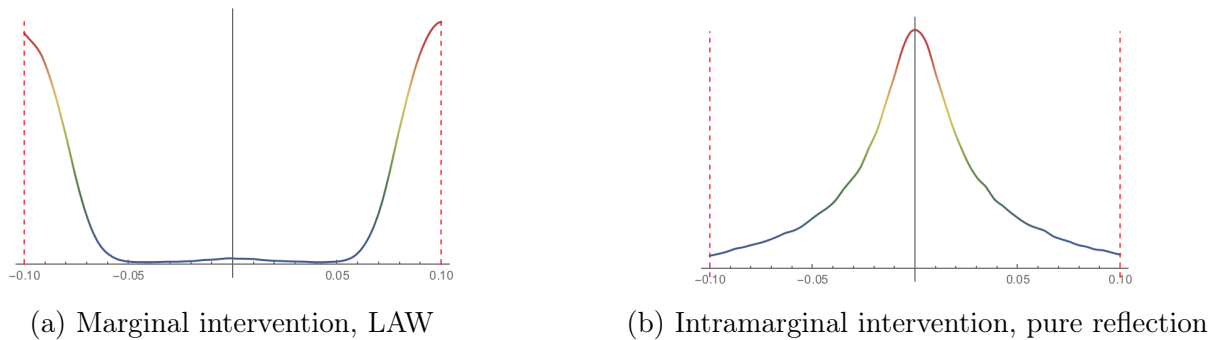
where  $\pi_{\partial D}(x)$  is the projection of  $x$  on the boundary  $\partial D$  parallel to the intervention  $\gamma$ . If  $F_{t+1}^{N,i} \in \partial D$ , then obviously  $f_{t+1}^N = F_{t+1}^{N,i}$ . For more references, see [Bossy et al. \(2004\)](#). The exchange rate path is then obtained simply by setting  $X_t^N = X^*(f_t^N, T - t)$  for every  $t \in [0, T]$ . It is of fundamental importance to set  $\Delta t$  equal to the update ratio given by  $1/\alpha$  in our model, so that the increment of the simulated exchange rate path has the same updating time frequency as the central bank. We can now discuss two kinds of interventions: the kind that intervenes by reflecting the process so that it just stays within the band (sometimes called “leaning against the wind”), and the pure reflection variety, which projects the fundamental process by an amount equivalent to how much the process would have surpassed the boundary. This distinction can also be understood as the amount of reserves the central bank has at its disposal in order to stabilize the fundamental process: the greater this quantity, the more likely it is that the intervention will be of the pure reflection type. We also assume that an intervention is effective instantaneously. This distinction also has important implications in the resulting exchange rate density: as shown in [Figure 6](#), given our characterization of risk, the greater the  $\beta$ , the earlier the central bank will have to intervene, given the fundamental’s increased tendency to escape towards the bands. We present five possible scenarios by estimating Monte Carlo densities of the simulated exchange rate process: the first two correspond to the Gaussian case, where  $\beta = 0$  with each of the two intervention strategies. The densities are obtained by Monte Carlo simulation of  $N$  sample paths, binning the data and limiting the bin size to zero to obtain the convolution density, then averaging over the  $N$  realizations and interpolating the resulting points. For all figures  $N$  is set to 5000,  $\sigma = 0.1$ ,  $r = 0.5$ ,  $\alpha = 200$ ,  $T = 3$  and the exchange rate target band to  $\pm 10\%$ . For more references on the method, see [Asmussen and Glynn \(2007\)](#). We obtain a realization path for each of the two and obtain both U-shaped (corresponding to the base Krugman case) and hump-shaped densities, corresponding to the [Dumas and Delgado \(1992\)](#) framework. The realized densities are plotted in [Figure 8](#). We then simulate the case in which  $\beta > 0$  but is not large enough to trigger the regime shift, each one with a different intervention strategy: in the marginal intervention case we obtain the two-regime density ( $\beta = 5$ ), as in the [Bessec \(2003\)](#) framework, and in the intramarginal one we obtain a hump-shaped distribution as for all intramarginal intervention frameworks. These results are shown in [Figure 9](#). Note that this is a consequence of our characterization of risk: the tendency  $\beta$  of the fundamental to hit the boundary generates the two-regime shape, since even in a marginal framework the central bank will already intervene when at a distance from the bands. Furthermore, this is the case in which *de facto* bands start to appear. Finally, we present a case in which  $\beta$  is large

enough ( $\beta = 50$ ) to trigger the regime shift, and the band in fact ceases to exist: the tendency to escape leads to the fundamental process constantly surpassing the boundary, honeymoon effects are impossible and pure reflection intervention concentrates most of the realizations around the initial level. This, as  $N \rightarrow \infty$ , generates a Dirac delta function around the initial value of the fundamental. This is displayed in Figure 10.

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**Figure 8** Exchange rate densities,  $\beta = 0$

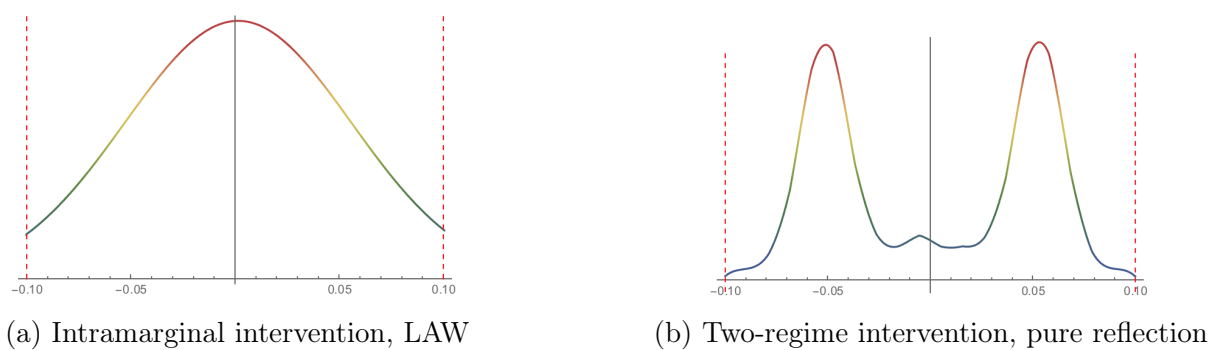
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**Figure 9** Exchange rate densities,  $\beta > 0, \beta < \beta^e$

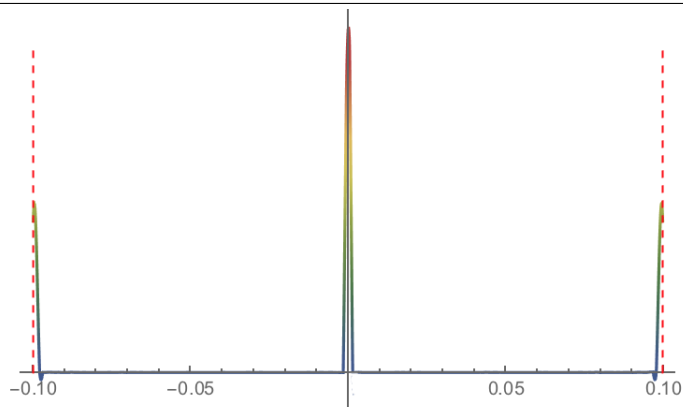
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**Figure 10** Exchange rate densities,  $\beta > \beta^e$ ,

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Target band too narrow given the level of underlying risk.

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## 8 Policy implications

A target zone with a terminal exit time to another currency has two objectives. First, the central bank wants to limit the volatility of its exchange rate ( $X_t$ ) versus the anchor currency below the free float level of the anchor currency ( $Z_t$ ). This provides us with a natural limiting condition

to the size of the band that the central bank can set:

$$|\bar{f}| \leq \sigma_z.$$

This essentially means that the band size of ERM-II of  $\pm 15\%$  will never be breached if the central bank of the target zone currency pegs to the Euro, as the Euro itself has an annualised volatility versus other major currencies in the range of 7-10%. Second, the central bank needs its target-zone to be considered feasible, in order to enjoy “honeymoon effects”, which in turn reduces the cost of intervention for achieving the set parity. In our setup, we propose the concept of a characteristic relaxation time  $\tau_{relax}$  which determines the minimum time a target zone must be maintained to “feel” the first effects of the home country central bank’s actions aimed at reducing fluctuations of the exchange rate, compared to a free float.<sup>10</sup> This allows us to interpret  $\tau_{relax}$  as the minimum time for agents to update their previously held exchange rate expectations, generating self-fulfilling expectations that create the honeymoon effect.

This does not mean that a central bank cannot adopt a target currency overnight with an arbitrary parity being the close of day value of the target exchange rate. In such a case, agents would not have had time to update their expectations and this would force the central bank to use a larger proportion of its assets (in the target currency) defending the parity level. This opens up many different avenues of enquiry into the expectation generation process of agents in foreign exchange markets. If  $t_{relax}$  is the minimum time for agents to update their previously held exchange rate expectations, this means that greater shifts in higher degree of agent risk-aversion (higher  $\beta$ ) will increase  $t_{relax}$ . As shown by [Osler \(1995\)](#) and [Lin \(2008\)](#), this effect would work through the feasibility of the target zone in time shifting speculators’ horizons towards short term speculation, where  $t_{speculation} \leq t_{relax}$ . This is a natural outcome of “honeymoon effects” which make intervention cheaper for central banks and harder for speculators after  $t_{relax}$ .

We find that  $t_{relax}$  is increasing with the magnitude of the risk aversion shifts, for  $\beta \leq \beta_e$ . Moreover, the target band size is also increasing in shifts size for  $\beta \leq \beta_e$  up to  $|\bar{f}| \leq \sigma_z$ .

New relevance for our framework has emerged in a recent development for the Economic Community of West African States (ECOWAS) as well as for new entrants in to ERM-II target zone. ECOWAS is planning to replace the current West African CFA Franc with a common currency, named Eco. The goal is for the 15 states to transition to the Eco via a target zone mechanism, similar to the ERM-II. Currently the CFA Franc is pegged to the Euro, with operational management shared between the Bank of France and the local central banks. After the reform, these countries would have to manage their own exchange rate targets without any outside support. The main political reason is understandable, and lies primarily in the severing of the ties with the former colonial ruler, France. The issues with this process, however, are multifaceted, and one of the main concerns is the short time horizon proposed for the target zone mechanism (one year). Moreover, there may be additional risk stemming from not allowing the ECB to have operational risk-sharing in the process, as well as the inherent risk faced by individual West African central banks. This translates directly to our framework, where the risk factor  $\beta$  may generate a relaxation time  $t_{relax}$  for individual states that may be larger than the proposed convergence time  $T$ . This could have potentially devastating consequences for the credibility of the participating central banks, and for the overall process of creating the new common currency. The inability of some ECOWAS countries to achieve the convergence criteria would make the adoption of the ECO impossible in the near future.

On the other hand, Bulgaria and Croatia officially entered ERM-II to replace their national currencies with the Euro in July 2020. The minimum convergence time  $T$  to exit to the Euro

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<sup>10</sup>The bank’s actions may be then viewed as a *de facto* reduction of the target zone band over time, whilst the *de jure* band remains unchanged.



is set at two years. Both the Lev and the Kuna have successfully pegged their currencies to the Euro over the last decade and may be considered as low  $\beta$  countries at the time of their entry into ERM-II. However, in with Fornaro (2020)’s predictions, the accession of these countries to the Euro will be followed by an upgrade in the country ratings for foreign currency debt.<sup>11</sup> These ratings upgrade at the time of the Covid-19 pandemic generates a higher probability of a capital flow surge and consequently a higher probability of a sudden stop into these countries. While capital flows might help with financing additional debt during the pandemic for these countries, it also generates a risk of these countries not meeting their fiscal criteria as well as destabilising the inflation expectations convergence process. It is highly unlikely given ECB support to the ERM process that these countries will not be able to successfully adopt the Euro. However, Lithuania’s experience in adopting the Euro provides a useful benchmark. These countries do run the risk of missing their fiscal and inflation criteria given the combination of push and pull factors, which are likely to generate high capital flows. If this risk materialises, this would imply that the Lev and the Kuna may have to wait in the ERM-II for a longer time than originally expected. This uncertainty is not as destabilising as in the case of the Eco, given ECB support. Nevertheless, there may be political consequences with support of the new currency if the ERM-II process goes on for longer than expected. An important future contribution of our work would be the structural estimation of the model parameters and an explicit computation of the relaxation time, thus effectively providing a lower bound for the necessary time for each country to reach the desired parity.

## 9 Conclusions

In this paper we have explored the implications of extending exchange rate target zone modeling to non-stationary dynamics and heavy, non-Gaussian tails stemming from time-varying investor risk aversion, which lead to mean-preserving risk increases in the fundamental distribution. Our framework leads to a natural interpretation of target zone feasibility, driven by the interplay between two contrasting forces: a destabilizing effect driven by risk which pushes the exchange rate towards the bands, and a stabilizing diffusive force.

Our model does not deal with optimal choices: indeed, the only choice variable potentially available to the authorities is the time horizon  $T$  by which the required parity needs to obtain. As such, from the policy perspective our model poses what is essentially a screening problem in the informational sense: in a worst case scenario, it is likely that neither of the two central banks knows the true riskiness of the fundamental process. If one chooses an exit time which is lower than the required minimum time at which parity can be reached (the relaxation time), the target zone exit time is not feasible. However, setting a  $T$  which is too high exposes one to increased business cycle risks, the dampening of which were a likely reason for entering a target zone in the first place. We show how our model effectively endogenizes the presence of the bands by the exchange rate expectations, and how the interplay between risk and target band has key implications in the credibility of the zone itself, as well as the possibility of honeymoon effects. Intervention is shown to be both marginal and intramarginal, depending on how much the central bank “hears” the distance to the target zone band. The potential emergence of regime shifts, furthermore, can further erode the target zone credibility. This allows the methods employed in this paper to be applied to a wide range of situations. An important future contribution of our work would be the structural estimation of the model parameters and an explicit computation of the relaxation time, thus effectively providing a lower bound for the necessary time for a country to reach the desired parity.

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<sup>11</sup>See <https://tinyurl.com/yyg4wp9p> and <https://tinyurl.com/yyuxm3tr> for more details on the potential for ratings upgrades due to ERM-II

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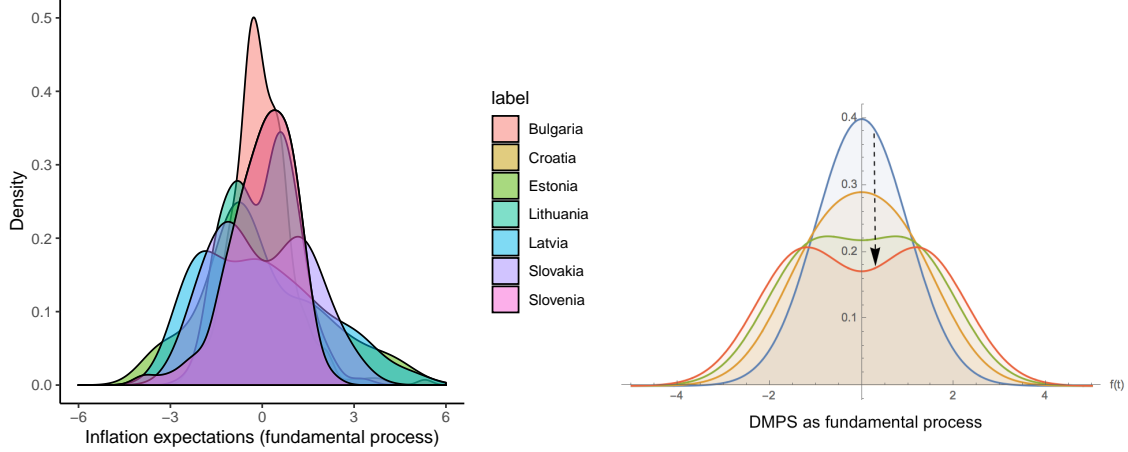
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# Appendices

## A Alternative interpretations of risk

**Figure 11** Estimated densities of the fundamental process (inflation expectations) for ERM-II currencies



(Left panel) Centered difference between Euro area inflation expectations and target zone country inflation expectations, for the time each currency was in the target zone with the Euro. Kolmogorov-Smirnov tests greatly reject each hypothesis of Gaussianity. The data for inflation expectations comes from the Euro Commission's Joint Harmonised Consumer Survey. For more details we refer to [Arioli et al. \(2017\)](#). Bulgaria and Croatia have only recently acceded to joining the Euro and the data for them is backward-looking to give the reader a sense of pre-target zone differences in inflation expectations. (Right panel) Transition densities of the fundamental process with mean-preserving spreads at time  $t = 1$ , each with risk increases in the direction of the arrow.

Destabilization is intrinsically connected to risk in the fundamental process. Besides the structural interpretation of risk as stemming from time-varying investor risk aversion, one could think of a variety of other interpretations for the parameter  $\lambda$  of increasing risk, which generates mean-preserving spreads in the density of the exchange rate fundamentals. A quick glance at the left-hand panel of Figure 11 shows that the difference in inflation expectations, one of the key fundamentals in the determination of exchange rate target zones, is undoubtedly non-Gaussian, exhibits substantially heavier tails and presents bimodal tendencies stemming from both inflationary and deflationary pressures shifting probability away from the center. Such risk dynamics cannot be represented by the variance of Gaussian fluctuations, as they cannot affect the distribution tails, but rather requires the presence of forces that increase the tendency of the fundamental process to escape its long-run level. The right panel of Figure 11 shows the transition density of the DMPS process at an arbitrary time for increasing risk. The DMPS density with  $\lambda$  parameter fit by maximum likelihood is a better fit for the empirical densities for each of the densities shown in 11.

Another way of interpreting the risk parameter of our framework could be via the presence of capital flows, especially in how the magnitude and the drivers of capital flows matter in determining the stabilisation effects. First, capital flows may be driven by push factors creating cycles of bonanzas and sudden stops seen with New Member States. [Hansson and Randveer \(2013\)](#) argue that capital flow dynamics were the key driver for cyclical developments in the Baltic ERM economies. This is might be a issue for a small target zone country if the capital flows generate excess appreciation or depreciation pressure weakening the feasibility of the target zone. This is particularly problematic if there is a sudden stop with reallocation of capital flows to more productive economies in the target zone as seen during the Eurozone crisis ([Ghosh et al., 2020](#)). Furthermore, assuming absence of macro-prudential tools, capital flow volatility may generate foreign exchange intervention volatility inside the target zone, as the use of interest rates as a monetary policy tool can generate further pro-cyclicality in capital flows. This nexus between capital flows and target zone management may destabilise the convergence in the inflation pro-

cess of the target zone country. This is the key source of additional risk in our setting. Let's consider the real interest rate version of the UIP condition:

$$\mathbb{E}\{dX_t\} = (r_t - r_t^*)dt + \mathbb{E}\{d\pi_t - d\pi_t^*\},$$

where  $\pi^*$  is the target country's inflation measure and  $\pi$  is home inflation. If there are high capital inflows that need to be counteracted by (unsterilised) intervention, this would generate a lower real interest rate of financing by putting downward pressure on  $r_t$ . This additional supply of credit is likely to increase the  $\mathbb{E}\{d\pi_t\}$ . This would require an interest rate response by the national central bank, in the absence of macro-prudential tools. We can see that in this particular case, increasing interest rates may be pro-cyclical to capital flows as long as the inflation process responds positively to the interest rate hike, causing a loss of monetary autonomy if the process is self-reinforcing. A destabilizing outcome of this setting would be if the inflation process does not respond to the interest rate moves and causes an outflow of capital flows. This would jeopardise the feasibility of the target zone and could cause the gap between  $r_t$  and  $r_t^*$  to become larger than before entering the target zone. The standard approach of modeling risk in the target zone does not consider the risk stemming from the currency union itself. If the target currency union has real interest rate changes through lower expected inflation surprises, it will also affect the stability of target zone by the capital flow mechanism we have described.<sup>12</sup>

This shows that one cannot conflate all information pertaining to risk with the variance parameter of the Gaussian distribution. We therefore adopt a definition of risk which corresponds to the concept of a mean-preserving increase in risk, in the second-order stochastic dominance sense, often referred to as a mean-preserving spread. Risk forces are by construction characterized by a second-order stochastic dominance criterion, and therefore must leave the long-run level unchanged. We therefore need to characterize such forces as mean-preserving increases in risk. This concept was introduced in a static setting in the seminal contributions of [Rothschild and Stiglitz \(1970 and 1971\)](#), who define two sufficient integral conditions that allow one to unambiguously rank distributions in terms of their riskiness, and extended to a dynamic setting by [Arcand et al. \(2020\)](#). This rigorous characterisation of risk is not considered by the extant target zone literature. Lastly, we note that our characterization of risk as destabilizations caused by capital flows can be further extended to any source of external risk, and our model framework would still apply.

## B Derivations of the stationary and transient equations

For the derivation of the stationary solution, we first introduce the following integral transformation:

$$X(t, f) = \int^f \cosh(\beta\zeta) Y(t, \zeta) d\zeta \quad \Longleftrightarrow \quad \partial_f X(t, f) = \cosh(\beta f) Y(t, f), \quad (21)$$

which is Darboux-type functional transformation. As shown in [Arcand et al. \(2020\)](#), Eq.(22) can be written equivalently as

$$\partial_t X(t, f) + \frac{\sigma^2}{2} \partial_{ff} X(t, f) + \beta \tanh(\beta f) \partial_f X(t, f) - \alpha X(t, f) = -\alpha f. \quad (22)$$

which leads to:

$$\partial_t Y(t, f) + \frac{\sigma^2}{2} \partial_{ff} Y(t, f) - \left[ \frac{\beta^2}{2} + \alpha \right] Y(t, f) = -\alpha f \cosh(\beta f). \quad (23)$$

Setting  $\partial_t = 0$  one obtains a nonlinear ODE in  $f$  which has the closed form solution as given by (12), which is the sum of the general solution (two opposite-sided exponentials) and a particular solution. Inverting the transformation back to  $X$  one obtains (11).

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<sup>12</sup>For simplicity, we do not consider the currency union having positive inflation surprises, even though in a low real interest rate setting, it may lead to capital flows to the target zone currency. This mechanism can be amplified by presence of multiple currencies in the target zone with cross-currency constraints on movement versus the target currency ([Serrat, 2000](#)).

For the transient dynamics, we need to solve the following equation:

$$\begin{cases} \partial_\tau X(\tau, f) - \frac{\sigma^2}{2} \partial_{ff} X(\tau, f) - \beta \tanh(\beta f) \partial_f X(\tau, f) + \alpha X(\tau, f) = +\alpha f, \\ X(0, f) = 0. \end{cases} \quad (24)$$

Writing  $X(\tau, f) = X^*(\tau, f) + X_S(f)$ , Eq.(24) implies:

$$\begin{cases} -\frac{\sigma^2}{2} \partial_{ff} X_S(f) - \beta \tanh(\beta f) \partial_f X_S(f) + \alpha X_S(f) = +\alpha f, \\ \partial_\tau X^*(\tau, f) - \frac{\sigma^2}{2} \partial_{ff} X^*(\tau, f) - \beta \tanh(\beta f) \partial_f X^*(\tau, f) + \alpha X^*(\tau, f) = 0. \end{cases} \quad (25)$$

While the first line in Eq.(25) has already being solved in Eq.(11), the second line needs now to be discussed. Writing again  $X^*(\tau, f) \cosh(\beta f) := Y^*(\tau, f)$ , we obtain:

$$\partial_\tau Y^*(\tau, f) - \frac{\sigma^2}{2} \partial_{ff} Y^*(\tau, f) + \left[ \frac{\beta^2}{2} + \alpha \right] Y^*(\tau, f) = 0. \quad (26)$$

The smooth-pasting conditions given by Eq.(13) imposes:

$$\begin{cases} \partial_f X^*(\tau, f) |_{f=\underline{f}} = 0 & \Rightarrow \quad \{[\partial_f Y^*(\tau, f)] - \beta \tanh(\beta f) Y^*(\tau, f)\} |_{f=\underline{f}} = 0, \\ \partial_f X^*(\tau, f) |_{f=\bar{f}} = 0 & \Rightarrow \quad \{[\partial_f Y^*(\tau, f)] - \beta \tanh(\beta f) Y^*(\tau, f)\} |_{f=\bar{f}} = 0. \end{cases} \quad (27)$$

We solve (26) by separation of variables and expansion over the basis of a complete set of orthogonal eigenfunctions. The solution can be expressed as  $Y^*(\tau, f) = \phi(\tau)\psi(f)$ , and therefore we can write it as

$$\frac{\dot{\phi}(\tau)}{\phi(\tau)} = \lambda_k = \frac{\sigma^2}{2} \frac{\psi''(f)}{\psi(f)} - \rho$$

where  $\rho = \left[ \frac{\beta^2}{2} + \alpha \right]$ .

The time-dependent part solves to  $\psi(\tau) = \exp(\tau \lambda_k)$ , and the fundamental-dependent part can be written as

$$\psi''(f) - 2(\lambda_k + \rho)\psi(f) = \psi''(f) + 2\frac{\Omega_k^2}{\sigma^2}\psi(f) = 0.$$

The rest of the derivations follow straightforwardly, solving for  $\psi$  and obtaining the eigenfunctions

$$\psi_k(f) = c_1 \cos\left(\sqrt{2}\frac{\Omega_k}{\sigma} f_t\right) + c_2 \sin\left(\sqrt{2}\frac{\Omega_k}{\sigma} f_t\right).$$

which form an orthogonal basis for the space of  $2\bar{f}$  well-behaving functions. Smooth-pasting conditions impose  $c_1 = 0, c_2 = 1$  and we obtain the form of the eigenfunctions as given by (16). The Fourier coefficients follow in their standard form, using the stationary equation  $X_S(f_t)$ .

## C Risk, regime shifts and honeymoon effects

We now briefly discuss the connection between risk and the honeymoon effect, and how such effects cannot be obtained when the destabilizing effects of risk shocks in the fundamental are too strong. For illustrative purposes, let us consider a baseline case of our model in a symmetric band  $[-\bar{f}, \bar{f}]$  around the parity 0, and let us compare the DMPS-driven model with the standard Gaussian one. Omitting time dependency, we have again the framework given by



$$X = f + \frac{1}{\alpha} \frac{\mathbb{E}\{dX\}}{dt},$$

which leads to the following couple of PDEs, depending on the form of the fundamental process.

$$\begin{cases} X = f + \frac{1}{2} \partial_{ff}[X(f)] & \text{(Gaussian),} \\ X = f + \frac{1}{2} \partial_{ff}[X(f)] + \beta \tanh(\beta f) \partial_f[X(f)] & \text{(DMPS).} \end{cases}$$

We now focus on the stationary regime for which get the general solutions:

$$\begin{cases} X(f) = f + A_0 \sinh(\rho_0 f), & \text{(Gaussian),} \\ X(f) = f + A_\beta \frac{\sinh(\rho_\beta f)}{\cosh(\beta f)}, & \text{(DMPS),} \end{cases}$$

where  $\rho_\beta = \sqrt{\beta^2 + 4\alpha}$  and  $A_\beta$  is a yet undetermined amplitude. We now apply the smooth fitting procedure at the target level  $\bar{f}$ <sup>13</sup>.

For the standard Gaussian framework we have  $X(f) \mapsto X_0(f) = f + a \sinh(\rho_0 f)$ , since  $\beta = 0$  and consequently  $\rho \mapsto \rho_0 := \sqrt{\frac{2\alpha}{\sigma^2}}$ . We therefore have :

$$X_0(f) = a \tanh(\rho_0 f) + f, \quad \rho_0 = \sqrt{\frac{2\alpha}{\sigma^2}},$$

which is the same result as in the standard Gaussian models. In particular, denote  $W$  the contact point with the target boundary  $\pm \bar{f}$ , we have

$$\begin{cases} \bar{f} = X_0(W) & \Rightarrow & F = W + a \tanh(\rho_0 W), \\ 0 = 1 + \rho_0 a \cosh(\rho_0 W) & \text{(smooth fitting at position } W). \end{cases} \quad (28)$$

From the second line in Eq.(28), we conclude immediately that:

$$a = \frac{-1}{\rho_0 \cosh(\rho_0 W)}.$$

and accordingly, we end with:

$$X_0(f) = f - \frac{\sinh(\rho_0 f)}{\rho_0 \cosh(\rho_0 W)} \quad (29)$$

Furthermore, we can verify that  $W \in \mathbb{R}^+$  for all values of the parameter  $\rho_0 > 0$ . Eq.(29) implies that:

$$W - F = \frac{\tanh(\rho_0 W)}{\rho_0}.$$

It's immediately seen that the last equation always possesses a single solution  $W > 0$ . Let us now examine the paper's main framework, the case where  $\beta > 0$ . In this case, for a target zone with band size  $F$  and a smooth contact point  $W$ , we have:

$$\begin{cases} F = W + a \frac{\sinh(\rho W)}{\cosh(\beta W)} + \omega \tanh(\beta W) \\ 0 = 1 + \frac{a}{\cosh(\beta W)} [\rho \cosh(\rho W) - \beta \sinh(\rho W) \tanh(\beta W)] + \frac{\omega \beta}{\cosh^2(\beta W)}. \end{cases} \quad (30)$$

The second line of the last equation implies:

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<sup>13</sup>Due to the symmetry, we have here only one amplitude  $A$  to determine since only one boundary needs to be considered.

$$a = -\frac{\cosh^2(\beta W) + \beta \omega}{\cosh(\beta W) \cosh(\rho W) \underbrace{[\rho \tanh(\rho W) - \beta \tanh(\beta W)]}_{:=\Delta}}.$$

From the last line, let us consider the equation  $\Delta = 0$ . First we remember from the very definition that  $\rho \geq \beta$  and hence the equation:

$$\frac{\rho}{\beta} \tanh(\rho W) = \tanh(\beta W) \quad \Leftrightarrow \quad \Delta = 0.$$

Since  $\frac{\rho}{\beta} > 1$  the last equation has necessarily a solution which is denoted  $\pm W_c$ . Note in addition that for a couple of  $\beta$  such that  $\beta_1 < \beta_2$ , we have:

$$\beta_1 > \beta_2 \quad \Leftrightarrow \quad W_{c,1} < W_{c,2} \quad (31)$$

and for  $\beta \rightarrow \infty$ , we shall have  $W_c \rightarrow 0$ . Now from  $W$  solving the first line of Eq.(30), we may have the alternatives:

$$\begin{aligned} a) \quad & W_c < W, \\ b) \quad & W_c \geq W. \end{aligned} \quad (32)$$

For case a), standard boundary techniques cannot be applied as in the Gaussian case, and hence the limit  $W = W_c$  explains the regime transition observed in the spectrum. This is due to the fact that for large  $\beta$  the honeymoon effect range becomes effectively large enough to preclude the possible existence of a target zone.

## D Noise sources driving the fundamental

Let us now assume that the fundamental is driven by a couple of noise sources, namely i) composite shocks  $v_t$  and ii) fluctuations in the money supply  $m_t$ , given by Gaussian noise around a drift  $\mu$ . We therefore add another source of noise, but we are not necessarily increasing the risk in the fundamental process. We then have

$$\begin{cases} df_t = \sigma_1 dW_{1,t} + dm_t, \\ dm_t = \mu dt + \sigma_2 dW_{2,t}, \quad m_{t=0} = m_0. \end{cases} \quad (33)$$

where the noise sources  $dW_{1,t}$  and  $dW_{2,t}$  are two independent White Gaussian Noise (WGN) processes. We then obtain  $f_t$  as a Gaussian process, since trivially

$$df_t = \mu dt + \sqrt{\sigma_1^2 + \sigma_2^2} dW_t$$

and we are exactly in the standard framework (in the literature usually  $\mu = 0$ ), only with a change in variance. If however we wish to incorporate a general increase in risk, and one that may represent the force that was discussed in Section 2, we can write the following more general framework:

$$\begin{cases} df_t = \sigma_1 dW_{1,t} + dm_t, \\ dz_{\beta,t} = \zeta(\beta; z_t) dt + \sigma_2(\beta) dW_{2,t}, \quad z_{t=0} = 0. \end{cases}$$

where  $\beta \geq 0$  is a control parameter and the repulsive drift  $\zeta(\beta; z) = -\zeta(\beta; -z) < 0$  models an extra risk source via a dynamic zero mean process. We parametrize risk with  $\beta$ , and therefore  $\beta = 0$  simply implies  $\sigma_2(\beta) = \zeta(0; z_t) = 0$  implying that the process is Gaussian and driven entirely by the composite shock process. Our candidate for  $\zeta$  is the DMPS process:

$$df_t = \sigma_1 dW_{1,t} + dz_t = \beta \tanh(\beta z_t) dt + \sigma_1 dW_{1,t} + \sigma_2(\beta) dW_{2,t}$$

$\Downarrow$

$$dz_t = \beta \tanh(\beta z_t) dt + \left[ \sqrt{\sigma_1^2 + \sigma_2^2(\beta)} \right] dW_t, \quad z_{t=0} = 0.$$

where we used the fact that the difference between two independent WGN's is again a WGN with variance as given in the previous equation. Alternatively one may formally write:

$$\begin{aligned} df_t &= \sigma_1 dW_{1,t} + \beta \tanh \left[ \beta \overbrace{(f_t - \sigma_1 W_{1,t})}^{z_t} \right] dt + \sigma_2(\beta) dW_{2,t} = \\ &\quad \beta \tanh \left[ \beta \overbrace{(f_t - \sigma_1 W_{1,t})}^{z_t} \right] dt + \left[ \sqrt{\sigma_1^2 + \sigma_2^2(\beta)} \right] dW_t, \end{aligned}$$

Using the initial equation (8) and the previous equation and applying Itô's lemma to the functional  $X(f_t, t)$ , we obtain:

$$\begin{aligned} \frac{(1-r)}{\alpha} \left\{ \partial_t X(f, t) + \partial_f X(f, t) \underbrace{\mathbb{E} \{ \beta \tanh [\beta (f_t - \sigma_1 W_{1,t})] \}}_{= \beta \tanh[\beta(f)]} + [\sigma_1^2 + \sigma_2^2(\beta)] \partial_{ff} X(f, t) \right\} = \\ X_t - r f_t \end{aligned} \quad (34)$$

In the last Eq.(34), the under-brace equality follows since all odd moments in the expansion of the hyperbolic tangent vanish and the  $\tanh(x)$  is itself an odd function. Now, normalizing as to have  $[\sigma_1^2 + \sigma_2^2(\beta)] = 1$ , we are in the nominal setting of our paper.

## E Attracting drift: mean-reverting dynamics

A fully similar discussion can be done for mean-reverting fundamental dynamics (Ornstein-Uhlenbeck dynamics) reflected inside an interval  $[\underline{f}, \bar{f}]$ . In this section, the fundamental is driven by the mean-reverting dynamics:

$$df = \lambda(\mu - f)dt + \sigma dW_t,$$

where  $\mu$  is the “long-run” level of the fundamental, and  $\lambda$  is now the speed of convergence, to highlight the mean-reverting equivalent of the DMPS process. Following the previous exposition, we can obtain the full solution for the exchange rate  $X^*(t, f)$  as the solution of

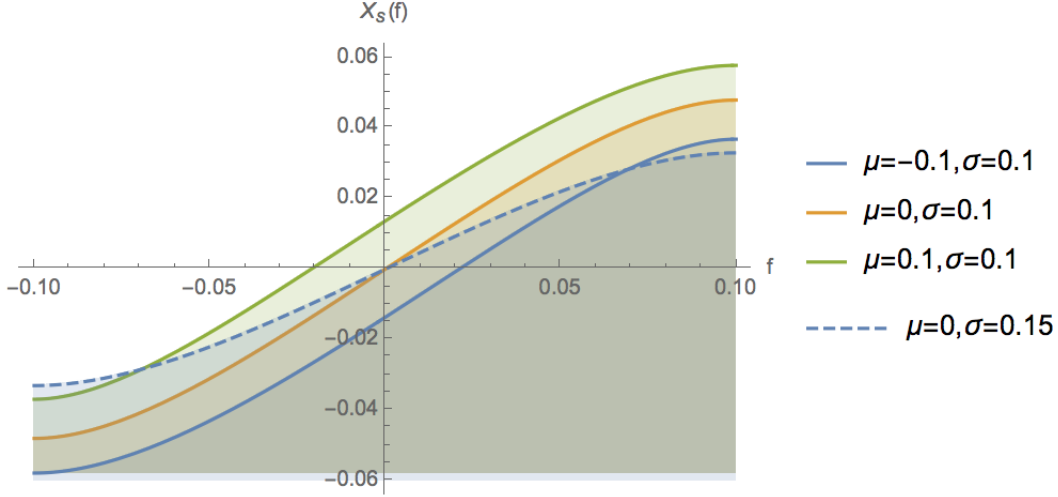
$$\partial_t X + \frac{\sigma^2}{2} \partial_{ff} X + \lambda(\mu - f) \partial_f X - \frac{\alpha}{1-r} X = -\frac{r\alpha}{1-r} f.$$

As before, we have the stationary solution for a vanishing  $\partial_t$ , and here it reads

$$\begin{aligned} X_S(f) &= A {}_1F_1 \left[ \frac{\alpha}{2\lambda(1-r)}, \frac{1}{2}; \frac{\lambda}{\sigma^2} (f - \mu)^2 \right] + \\ &+ B \frac{\sqrt{\lambda}}{\sigma} (f - \mu) {}_1F_1 \left[ \frac{\alpha}{2\lambda(1-r)} + \frac{1}{2}, \frac{3}{2}; \frac{\lambda}{\sigma^2} (f - \mu)^2 \right] + \\ &+ \left[ \frac{\lambda\mu(1-r)f + r\alpha}{\lambda(1-r) + \alpha} \right] \end{aligned} \quad (35)$$

where  ${}_1F_1[a, b; x]$  is the confluent hypergeometric function. The integration constants  $A$  and  $B$ , as before, are determined via smooth pasting at the target zone boundaries, namely:  $\partial X_S(f)|_{f=\underline{f}} = \partial X_S(f)|_{f=\bar{f}} = 0$ . Note that if  $\mu = 0$ , then  $A = 0$ . Figure 12 shows the stationary dynamics of the exchange rate as

**Figure 12** Mean-Reverting Stationary Dynamics



function of the fundamental, for different values of long-run level  $\mu$  and noise variance  $\sigma$ . The band is assumed symmetric around 0, and  $\bar{f} = 10\%$ .

The associated Sturm-Liouville equation is now given by

$$\frac{\sigma^2}{2} \partial_{ff} X + \lambda(\mu - f) \partial_f X + \rho X = 0,$$

where  $\rho = \frac{\alpha}{1-r}$ , and the spectrum of the process can be obtained explicitly by solving a transcendental equation involving Weber parabolic cylinder functions. As before, the complete solution is given by an expansion on a complete set of orthogonal functions on the target band, namely:

$$X^*(T-t, f) = X_S(f) + \sum_{k=1}^{\infty} c_k \exp[-(\Omega_k + \rho)(T-t)] \psi(\Omega_k, f),$$

where the Fourier coefficients  $c_k$  again impose the terminal condition  $X^*(0, f) = -X_S(f)$ . As worked out by Linetsky (2005) explicit though lengthy closed form expressions are obtainable (see Eqs.(39) and (40). For the case of a symmetric target zone  $\underline{f} = -\bar{f}$ , an approximation valid for large eigenvalues  $\Omega_k$ , (i.e. large  $k$ 's) is given in [L] and reads:

$$\begin{aligned} \Omega_k &= \frac{k^2 \pi \sigma^2}{8 \bar{f}^2} + \frac{\lambda}{2} + c_0 + O\left(\frac{1}{k^2}\right) \\ c_0 &= \frac{\lambda^2}{6 \sigma^2} (4 \bar{f}^2 - 6 \bar{f} \mu + 3 \mu^2). \end{aligned} \quad (36)$$

The normalised eigenfunctions, also up to  $O\left(\frac{1}{k^2}\right)$ , read:

$$\begin{aligned} \psi_k(f) &= \pm \frac{\sigma}{\sqrt{2}} \bar{f}^{-1/2} \exp\left[\frac{\lambda(f-\mu)^2}{2\sigma^2}\right] \left[ \cos\left(\frac{k\pi f}{2\bar{f}}\right) + \frac{2\bar{f}}{k\pi\sigma^2} \phi(f) \sin\left(\frac{k\pi f}{2\bar{f}}\right) \right] \\ \phi(f) &= \frac{\lambda^2}{6\sigma^2} f^3 - \frac{\lambda^2 \mu}{2\sigma^2} f^2 - \left[ \frac{\lambda}{2} \left( \frac{\sqrt{2\lambda}}{\sigma} \mu + 1 + c_0 \right) \right] f + \theta \mu \end{aligned} \quad (37)$$

While strictly speaking Eq.(36) furnishes very good estimates for large  $k$  values, a closer look in [L] shows that even for low  $k$ 's, ( $k = 1, 2, \dots$ ), pretty good approximations are also obtainable. In particular, for

$k = 1$ , we approximately have:

$$\tau_{\text{relax}} \simeq [\Omega_1]^{-1} = \left[ \frac{\pi \sigma^2}{8 \bar{f}^2} + \frac{\lambda}{2} + c_0 \right]^{-1}.$$

For this mean-reverting dynamics, the interplay between risk (here solely due to the noise source variance  $\sigma^2$ ) and the target band width  $2\bar{f}$  on  $t_{\text{relax}}$  is opposite compared to the DMPS dynamics of section 2.

The tendency of the fundamental  $f$  to revert to its long-run level  $\mu$ , for a narrow target band, generates an effect of an increase in risk (variance) that is opposite of the one generated by an increase of  $\beta$  in the DMPS setting, because of the latter's tendency to escape from the mean. If the band is larger, lower levels of  $\sigma$  initially increase the relaxation time, to ultimately achieving a decreasing effect. In both cases, an increase in the size of the target band requires a higher  $T$  in order for the target zone to be feasible.

We lastly notice that for the O-U case, zero is always the first eigenvalue (not surprising, given that it's an ergodic process) and a regime shift cannot be possible.

## F Alternative to O-U dynamics: softly attractive drift

We now present the model where we model the fundamental as an ergodic process with a softly attractive drift instead of the Ornstein-Uhlenbeck dynamics. This framework has the advantage of incorporating mean-reverting dynamics while retaining analytical tractability. By “softly attractive” drift we mean the DMPS drift with opposite sign, i.e.  $-\beta \tanh(\beta f)$ . This model presents similar dynamics to the O-U framework, and allows for a stationary time-independent probability measure. The marginal difference with the O-U advantage is that the reversion of the fundamental to the mean is softer, and the advantage is that the full spectrum is available and the dynamics do not require an approximation. The equation for the exchange rate after applying Itô's lemma is now given by

$$\partial_t X(t, f) + \frac{1}{2} \partial_{ff} X(t, f) - \beta \tanh(\beta f) \partial_f X(t, f) - \alpha X(t, f) = -\alpha f. \quad (38)$$

Using the equivalent transformation as in the DMPS case, we plug in Eq.(38) into Eq.(21) and obtain:

$$\begin{aligned} & \int^f \cosh(\beta \zeta) \partial_t Y(t, \zeta) + \\ & \frac{1}{2} [\beta \sinh(\beta f) Y(t, f) + \cosh(\beta f) \partial_f Y(t, f)] - \\ & \beta \sinh(\beta f) Y(t, f) - \alpha \int^f \cosh(\beta \zeta) Y(t, \zeta) d\zeta = -\alpha f. \end{aligned} \quad (39)$$

Now, taking once more the derivative of Eq.(39) with respect to  $f$ , one obtains:

$$\partial_t Y(t, f) + \frac{1}{2} \partial_{ff} Y(t, f) - \left[ \frac{\beta^2}{2} + \alpha \right] Y(t, f) = -\alpha \frac{f}{\cosh(\beta f)}. \quad (40)$$

Observe now that Eq.(40) is once again equivalent to the standard BM motion case and we can repeat the same procedure we . The spectrum will now include the eigenvalue zero since we deal with a stationary case.

We now proceed as before and Eq.(40) reads:

$$-\partial_\tau Y(\tau, f) + \frac{1}{2} \partial_{ff} Y(\tau, f) - \left[ \frac{\beta^2}{2} + \alpha \right] Y(\tau, f) = -\alpha \frac{f}{\cosh(\beta f)}. \quad (41)$$

Consider now the homogenous part of Eq.(41), namely:

$$-\partial_\tau Y(\tau, f) + \frac{1}{2} \partial_{ff} Y(\tau, f) - \left[ \frac{\beta^2}{2} + \alpha \right] Y(\tau, f) = 0.$$

As done before, the method of separation of variables leads us to introduce  $Y(\tau, f) = \phi(\tau)\psi(f)$  and the previous equation can be rewritten as:

$$\frac{-\partial_\tau \psi(\tau)}{\psi(\tau)} + \frac{1}{2} \frac{\partial_{ff} \psi(f)}{\psi(f)} - \left[ \frac{\beta^2}{2} + \alpha \right] = 0.$$

and therefore we can write:

$$\begin{cases} \frac{-\partial_\tau \psi(\tau)}{\psi(\tau)} = \lambda_k, \\ \frac{1}{2} \frac{\partial_{ff} \psi(f)}{\psi(f)} - \left[ \frac{\beta^2}{2} + \alpha \right] = \lambda_k \end{cases}$$

Defining  $\Omega_k^2 = \left[ \frac{\beta^2}{2} + \alpha \right] + \lambda_k$ , the relevant eigenfunctions reads:

$$\psi(f) = c_1 \sin(\sqrt{2}\Omega_k f) + c_2 \cos(\sqrt{2}\Omega_k f).$$

Going back to Eq.(21), the boundary conditions at the borders of the target zone  $\bar{f} = -\underline{f}$  reads:

$$\partial_f \left[ \int^f \cosh(\beta\zeta) \psi(\zeta) d\zeta \right] \Big|_{f=\bar{f}} = 0.$$

which implies that:

$$\cosh(\beta\bar{f})\psi(\bar{f}) \quad \Rightarrow \quad c_1 = 0 \quad \text{and} \quad \Omega_k = (2k+1) \frac{\pi}{2\sqrt{2}\bar{f}}. \quad (42)$$

We note that Eq.(42) implies :

$$\lambda_k = \frac{(2k+1)^2 \pi^2}{8\bar{f}^2} - \frac{\beta^2}{2} - \alpha \geq 0. \quad (43)$$

Lastly, as expected, for the soft attractive case we are able to derive the exact spectrum analytically and unlike the DMPS case, there is no spectral gap.